JUNIOR MATHEMATICS

BOOK TWO

BRESLICH

An Illustration of the Use of Geometrical Forms

JUNIOR MATHEMATICS

BOOK TWO

BY

udolph
ERNST R. BRESLICH
Assistant Professor of the Teaching of Mathematics,
The College of Education

and

Head of the Department of Mathematics,
The University High School

The University of Chicago

New York
THE MACMILLAN COMPANY
1925

Set up and electrotyped. Published July, 1925.

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This is the second volume of a series of textbooks on junior high-school mathematics. Like the first volume, it is organized on the principles that were stated in the preface of Book One.

Geometry is the basis of the first part of this course. By actual measurement formulas are developed for finding the areas of triangles, quadrilaterals, and the circle. In these formulas quadratic terms and polynomials make their appearance and some of the fundamental operations with integral numbers are taught. Then follows a study of the common solids which leads to algebraic terms and polynomials of the third degree. Algebra in the first half of this course is a tool subject. Algebraic notation is used because it is helpful and because it simplifies discussions of geometry.

In the second half of Book Two emphasis is transferred from geometry to algebra. Positive and negative numbers are introduced. Geometry is now used to illustrate and to make concrete the meaning of abstract algebraic concepts and operations.

Opportunities for arithmetical computation and for problem solving are offered everywhere to help the pupil attain a high degree of accuracy in arithmetical work.

There is an abundance of real problems which will make the subject seem worth while to the learner, but

the author has intentionally made use of many problems of the traditional type wherever they can be made helpful for the purpose of drill and practice.

The author expresses his appreciation and gratitude to Director Charles H. Judd, Professor H. C. Morrison, and Professor W. C. Reavis for inspiration, advice, and support during the time in which the material was tried out in the laboratory schools.

A grant from the Commonwealth Fund has aided the studies which contributed to the development of this course. E. R. Breslich.

CONTENTS

INTRODUCTION

The success of the seventh-year book on combination mathematics which was the first volume of the series of which this book is the second, supplies gratifying evidence that junior high schools of various types and teachers with different kinds of preparation can use this material to the great advantage of junior high school pupils.

The report of the National Committee on Mathematical Requirements recommended combination mathematics in the junior high school but left to later experimentation the formulation of details of such a course. The preparation of a course of the type called for had been under way in the schools of the School of Education of the University of Chicago for some time before the appearance of the report of the Committee on Mathematics Teaching. Mr. Breslich, who had charge of the course, had accordingly the advantage of years of experience and was able to furnish material refined by much use in the classroom.

When he published his first volume administrators and teachers in some quarters questioned whether schools which had up to that time followed the traditional program could adopt the new type of material without a long period of readjustment. The answer which experience has given to such questioning is most

promising for the progress of mathematics and for the evolution of the junior high school.

Experience shows that pupils who formerly were uninspired by the review and repetitious number drills which are traditional in seventh and eighth grades, get a new view of the meaning of precise quantitative thinking when they are introduced to the methods and materials of elementary geometry and algebra. They exhibit an interest in these new subjects which was lacking under the older program. They show an intellectual maturity and grasp of methods of thinking which are usually thought to be beyond pupils of this age.

All these facts lead the advocates of combination mathematics and of the junior high school to express with more assurance than ever before their confidence in the movement to which the course which is presented in the following pages is a contribution.

CHARLES H. JUDD.

CHAPTER I

AREA OF RECTANGLES AND SQUARES. MULTIPLICATION OF POLYNOMIALS. SQUARE ROOT

Measuring the Surface of a Rectangle

1. What we are going to study in this chapter. When buying land one should be able to determine its size, *i.e.*, to *measure* it. We have learned how to measure line segments, circle arcs, and angles. We know that in measuring a line segment, the segment is compared with a *unit segment*. We have measured angles

and arcs by means of *unit angles* and *unit arcs*. Since pieces of land, such as fields, lots, townships, and states, are very often in the form of polygons, we shall learn to *measure surfaces* (interior) of polygons. This means that we shall learn how to *determine the number*

of times a unit surface is contained in the surface of the polygon. The surface of a square is used as a *unit surface.*

We shall first see that the measure of surfaces of rectangles or squares may be found by counting the number of unit squares which they contain. Then a formula will be developed from which the result can be obtained more easily and accurately than by counting unit squares.

By the use of this formula, we shall solve some simple problems and equations, multiply a polynomial by another, such as $(a+b+c)$ by $(m+m)$, and find square roots of arithmetical numbers.

Finally we shall work out a relation from which one side of a right triangle can be found if the other two are known. For example, in the triangle ABC (Fig. 1), if a is the unknown length of the side CB, if c is the known length of the hypotenuse BA, and if b is the known length of AC, it will be shown that a can be found by solving the equation $a^2+b^2=c^2$ for a.

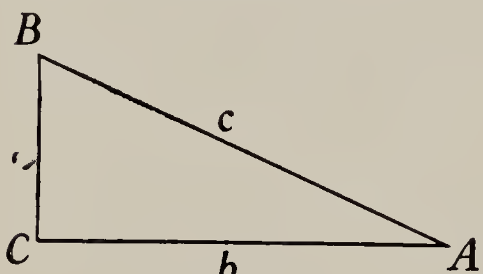


FIG. 1

We shall learn how to solve such equations.

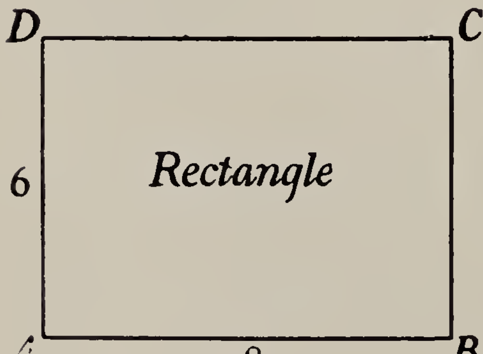


FIG. 2

2. Some important properties of a rectangle. A quadrilateral whose angles are right angles is a rectangle (Fig. 2). One of the two sides, as AB, is the *base* and the other, AD, is the *altitude.*

If the rectangle is *equilateral* it is a *square* (Fig. 3).

The following exercises establish two important properties of the rectangle:

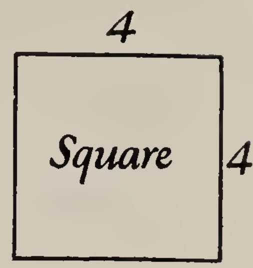


FIG. 3

EXERCISES

1. Draw a rectangle (Fig. 2), using ruler and protractor. Make the side AB equal to 8 centimeters, and AD equal to 6 centimeters.

2. Measure the other sides of the rectangle drawn in Exercise 1. Your results should show that *the opposite sides of a rectangle are equal.*

3. Measure the diagonals of the rectangle and show that *the diagonals of a rectangle are equal.*

3. How to measure the surface of a rectangle by counting unit squares. Let us measure the interior of a rectangle whose base AB is 3 centimeters long and whose altitude BC is 2 centimeters.

Place the rectangle on squared paper (Fig. 4).

Draw lines EF and GH, dividing the rectangle into three strips equal in size, like the strip $AEFD$. Divide each of the strips into two equal squares.

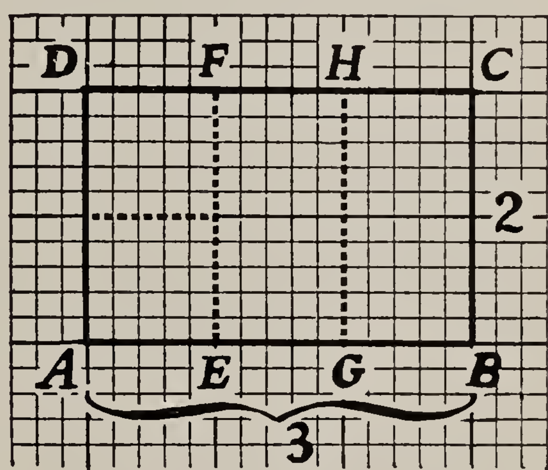


FIG. 4

Show that the rectangle is now divided into 3×2 equal squares.

Using one of these squares as a unit, we find that the measure of the rectangular surface $ABCD$ is equal to 3×2 square centimeters, or 6 square centimeters.

Find the measure of the surface $ABCD$, if one of the *small* squares is used as a unit.

4. What is meant by area. The number of times the surface of a unit square is contained in the interior of a polygon is the **area** of the polygon. Thus, the area of the rectangle shown in Fig. 4 is 6, or 150, according as a large or small square is used as a unit.

Exercises

1. On squared paper draw a rectangle whose base is 5 centimeters and whose altitude is 3 centimeters.

Find the area by counting unit squares.

Find the product of the lengths of the sides (dimensions) of the rectangle.

Compare the area with the product of the dimensions and state your result.

2. By counting unit squares find the area of a rectangle 4 centimeters long and 2 centimeters wide. How may the area be expressed in terms of the dimensions? In terms of the base and altitude?

3. Draw a rectangle whose dimensions are 4 centimeters and 3.5 centimeters. Count the unit squares and find the area by adding the unit squares and the fractional parts of unit squares. Show that the area is equal to the product of the dimensions.

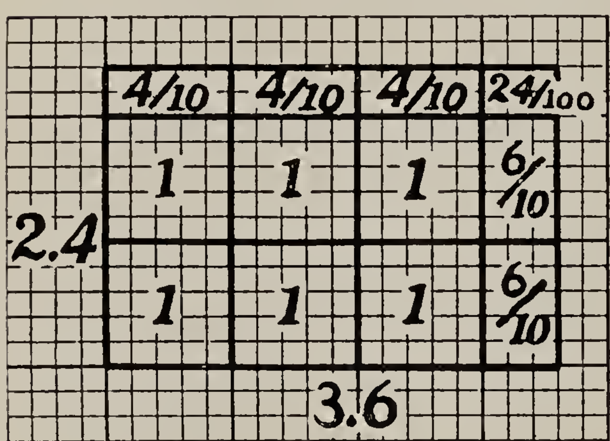


Fig. 5

4. Find the area of a rectangle 3.6 centimeters long and 2.4 centimeters wide.

Solution: Divide the interior of the rectangle as shown in Fig. 5. Show that the area is equal to

$$1+1+1+1+1+1+.6+.6+.4+.4+.4+\left(\frac{6}{10}\right)\times\left(\frac{4}{10}\right)$$

$$=6+2.4+.24=8.64.$$

Show that the same result may be obtained by multiplying 3.6 by 2.4.

5. Finding the area of a rectangle by means of a formula. The Exercises in §4 show that *the area of a rectangle is equal to the product of the base by the altitude.*

If we denote the lengths of the base and altitude by b and h, respectively, and the area by A, this principle may be translated into the formula $A = b \times h$. Writing briefly bh for $b \times h$ we have

$$\mathbf{A = bh.}$$

Exercises

In the following exercises arrange the written work as shown in Exercise 1.

1. Find the area of a rectangular flower bed 16 feet 6 inches long and 8 feet 8 inches wide.

Solution: $A = bh$. Why?
$b = 16\frac{1}{2}$. Why?
$h = 8\frac{2}{3}$. Why?

Hence $A = 16\frac{1}{2} \times 8\frac{2}{3} = \dfrac{\overset{11}{\cancel{33}} \times \overset{13}{\cancel{26}}}{\cancel{2} \times \cancel{3}} = 143$

$\therefore$ Area $= 143$ square feet.

2. Find the number of square feet of lighting surface in your classroom.

3. Find how much floor space per pupil your classroom contains.

4. Compare the lighting surface in your classroom with the floor space by finding the ratio of one to the other.

5. Find the number of square feet in the floor of a corridor 300 feet long and 12 feet wide.

6. A cement walk $2\frac{1}{2}$ feet wide and 120 feet long is to be laid for 30 cents a square foot. Find the amount paid for the walk.

7. A fence 8 feet high surrounds a rectangular field. The field is 250 yards by 300 yards. How much paint will be needed to paint the fence on the smooth side, if a gallon of paint covers 250 square feet?

8. The dimensions of the floor of a rectangular room are 11 feet and 10 feet. If the room is 9 feet high, how much will it cost to plaster the walls and the ceiling at a rate of 45 cents per square yard, not allowing any deductions for windows or doors?

9. A farmer has 16 yards of wire fencing with which to enclose a vegetable bed of rectangular shape. If the length is to be three times the width, what is the area?

10. How many rolls of wall paper are needed to paper a wall 12 feet wide and 8 feet high, if the paper to be used is 18 inches wide and if each roll contains 8 yards?

Suggestion: Find the number of strips in a roll and discard the piece left over.

11. During an anniversary rug sale, a department store advertised rugs as shown below:

Which rugs in the advertisement give the most floor space for the money? Arrange all your work as shown in the table below:

Dimensions	*Area in Square Feet*	*Cost*	*Cost per Square Unit*
I. 27 × 54	$10\frac{1}{8}$	\$6.50	\$0.64
36 × 63	. . . .	10.00	. . . .
$4\frac{1}{2} \times 7\frac{1}{2}$	. . . .	22.50	. . . .
6 × 9	. . . .	41.00	. . . .
$8\frac{1}{4} \times 10\frac{1}{2}$	. . . .	62.50	. . . .
9 × 12	. . . .	67.50	. . . .
II. $4\frac{1}{2} \times 6\frac{1}{2}$	. . . .	11.75	. . . .
6 × 9	. . . .	22.00	. . . .
$8\frac{1}{4} \times 10\frac{1}{2}$	. . . .	36.00	. . . .
9 × 12	. . . .	42.00	. . . .
III. 60 × 84	. . . .	20.00	. . . .
72 × 84	. . . .	22.50	. . . .

Find the areas of rectangles of the following dimensions. State the equations and arrange your work as shown in Exercise 1 above.

12. 42 and 17.

13. 3.4 and 6.

14. 8.2 and 1.42.

15. $9\frac{1}{2}$ and $14\frac{2}{3}$.

16. $17\frac{5}{6}$ and $123\frac{4}{7}$.

17. .06 and .24.

18. A farmer got 59 bushels of potatoes from a patch 9 rods by 11 rods. Find the yield per acre.

Suggestion: 1 acre = 160 square rods.

19. What is the price of a field 400 feet wide and 650 feet long if land is worth $80 an acre?

20. The area of a football field is 52,800 square feet, and the length is 330 feet. Find the width.

21. Draw a rectangle whose area is 6; 5; 4×2; xy; mn; ab. Find the area ab, when $a=2$, $b=7$; $a=1.4$, $b=3.6$; $a=3.16$, $b=7.84$.

22. Divide the shaded surfaces (Fig. 6) into rectangles and find the area of each.

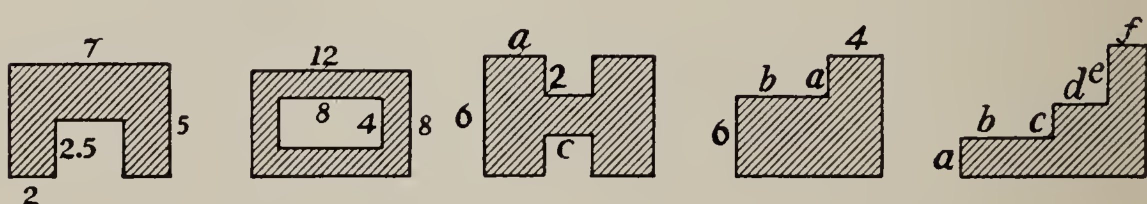


Fig. 6

Using the area sums obtained for the last three surfaces (Fig. 6) as formulas, find the area if $a=2$, $b=4$, $c=3$, $d=1.5$, $f=2\frac{1}{2}$; $a=3$, $b=7$, $c=2.5$, $d=4$, $e=4.5$, $f=3$.

Find the value of each of the polynomials in Exercises 23 to 30 if $a=6$, $b=2$, $c=1$, $d=\frac{1}{3}$.

23. $ab+ac+bc+4$.

Solution:
$$\begin{aligned} ab+ac+bc+4 &= 6\times2+6\times1+2\times1+4 \\ &= 12+6+2+4 \\ &= 24. \end{aligned}$$

24. $ac+bd+bc.$

25. $ab-bc+d+5.$

26. $\dfrac{ad+a-c}{ab}.$

27. $\dfrac{b+d+bd}{bd}$

28. $\dfrac{ab+ac}{ab-dc}.$

29. $\dfrac{bd+ba}{ad-c}.$

30. $\dfrac{ac-d}{ab-bc}.$

31. $\dfrac{2ab+c}{3ac+2d}.$

31. Draw a rectangle using protractor and ruler. Measure the sides and find the area.

32. Find the cost of covering a rectangle 25 inches by 18 inches with gold leaf at 9 cents a square inch.

33. A garden plot is 16 yards long and 18 yards wide. Find the cost of sodding it at 22 cents a square yard.

34. Find the cost of an inlaid floor 20 feet by 18 feet at $2.75 a square foot.

35. How many acres are there in a field 95 rods long and 72 rods wide?

6. **The area of a rectangle varies directly as the altitude.** Let $A=3h$ (§5) be the equation for finding the area A when b is equal to 3. Show that if $h=1$, $A=3$; if $h=2$, $A=6$; if $h=3$, $A=9$, etc. Thus when h is doubled, trebled, etc., we see that A is also doubled, trebled, etc. However, $\dfrac{A}{h}$ is always 3. Hence, A varies (changes), if h varies, but the ratio $\dfrac{A}{h}$ remains the same (constant).

Briefly we express this by saying that A *varies directly* as h.

7. Graphical representation of the formula $A = bh$. Let the altitude of a rectangle vary, the base remaining the same, *e.g.*, equal to 3. Show that when $h = 1, 2. 3,$ etc., $A = 3, 6, 9,$ etc.

Make a table of corresponding values of h and A (Fig. 7).

Make the graph of the equation $A = 3h$ as follows:

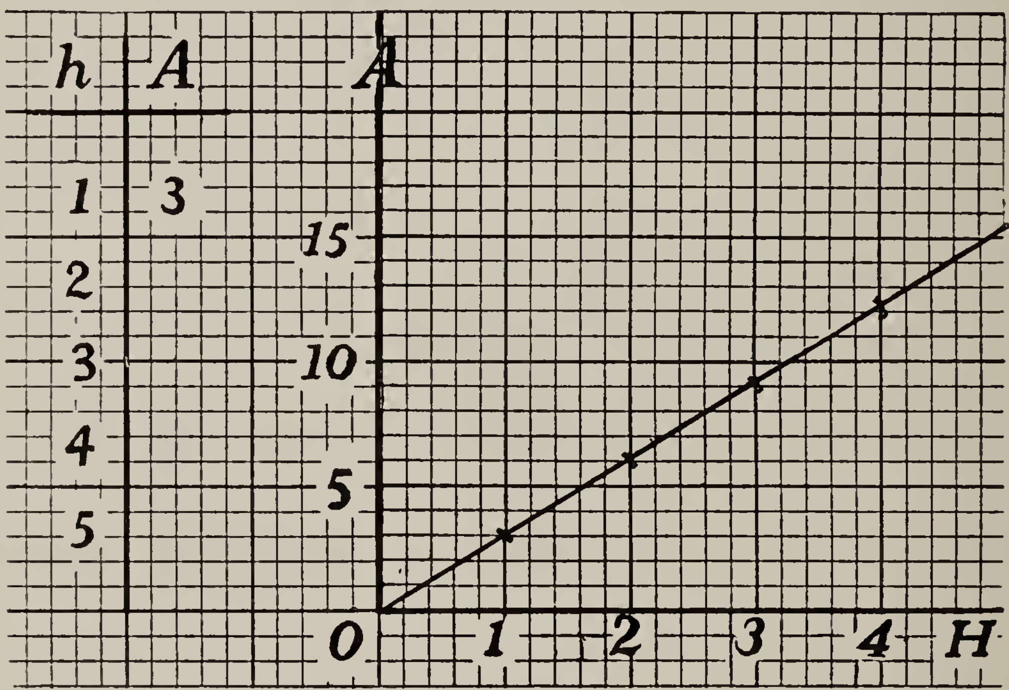


FIG. 7

On squared paper (Fig. 7) draw two lines OH and OA at right angles to each other.

On OH and OA mark off convenient units of length.

Plot the pairs of numbers in the table. Thus, the pair (1, 3) is plotted by passing from O to the right 1 unit and up 3 units.

Draw the line joining the points. This is the required graph.

EXERCISES

1. From the graph (Fig. 7) find A when $h = 1\frac{1}{2}$; 2.4; 3.8.
2. Make graphs of the equations $A = 5h$; $A = 8h$.

8. Using the rectangle to show the law of order of the factors of a product. We have seen that the area of a rectangle is equal to the product of the base by the altitude. Thus the area of $ABCD$ (Fig. 8) is 4×2.

Since a rectangle with a base equal to 2 and an altitude equal to 4 differs from the rectangle $ABCD$ only in position, its area, 2×4, must be equal to the area of $ABCD$, *i.e.*, equal to 4×2.

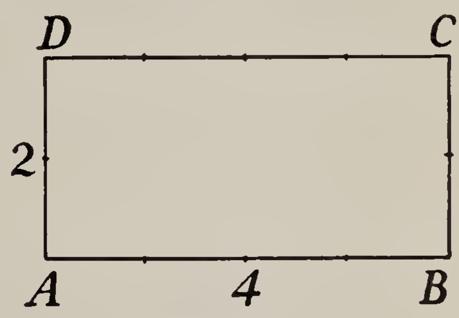


FIG. 8

This shows that $4\times2=2\times4$.

Any product, as xy, may be thought of as representing the area of a rectangle of dimensions x and y. Since this rectangle is equal to a rectangle of dimensions y and x, it follows that xy and yx have the same value. This illustrates the following principle, known as the *law of order in multiplication:*

The value of a product remains the same when the order of the factors is changed.

EXERCISES

1. Show from a drawing, as in Fig. 8, that $4\times6=6\times4$.

2. Show, by substituting values for a and b, that $ab=ba$.

Solution: LEFT SIDE		RIGHT SIDE
3×2		2×3
6	=	6

3. Explain why the formula $A=bh$ may be written $A=hb$.

4. By substituting values for x, y, and z show, as in Exercise 2, that $xy+yz+zx=yx+zy+xz$. For example, let $x=3$, $y=2$, $z=4$.

5. Show that $ab+bc+ca=ba+cb+ac$.

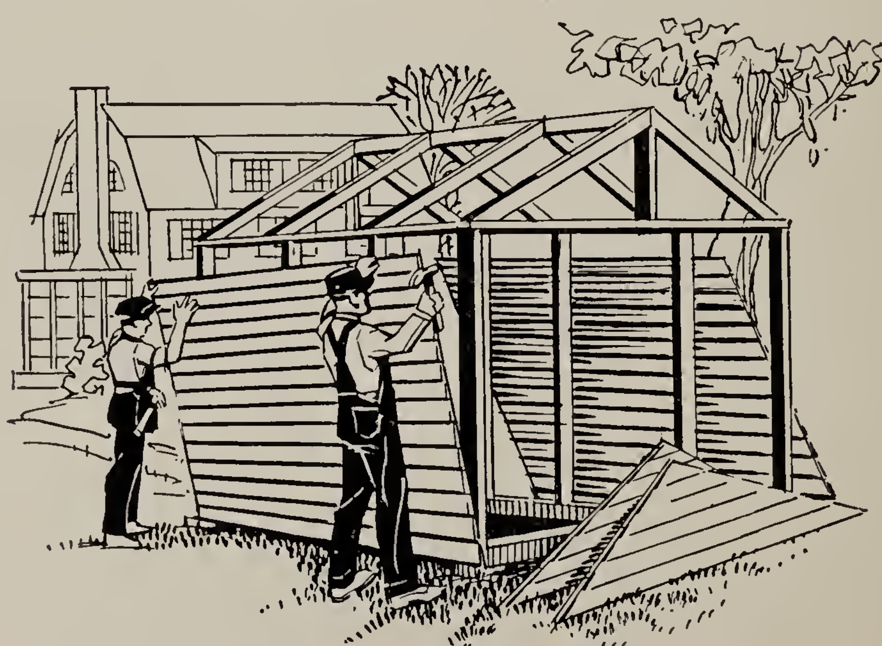
6. Find the value of $mn+nt+ts$ for $m=3.6$, $n=1.2$, $t=5.5$, $s=2.7$.

7. Since 4×0 means $0+0+0+0$, it follows that $4\times 0=0$. By means of the law of order show that the value of 0×4 is zero.

9. Multiplying numbers by zero. Exercise 7 (§8) illustrates that *the value of a product is zero if one of the factors is zero.*

Multiplication of a Polynomial by a Monomial

10. How to use the rectangle to multiply a polynomial by a monomial. A carpenter wishes to compute the number of square feet of siding needed for a garage whose floor is 18 feet$\times$20 feet, and whose walls are 9 feet high. To do this he may find the number of square feet needed for each side and then find the sum. This gives the number of square feet in the four walls equal to $(9\times 20)+(9\times 18)+(9\times 20)+(9\times 18)$.



The required number may also be found as follows: Imagine the 4 walls placed adjacent to each other (Fig. 9). They form a rectangle whose length is $20+18+20+18$, and whose width is 9. The number of square feet contained in the whole rectangle is $9\times(20+18+20+18)$, or 9×76.

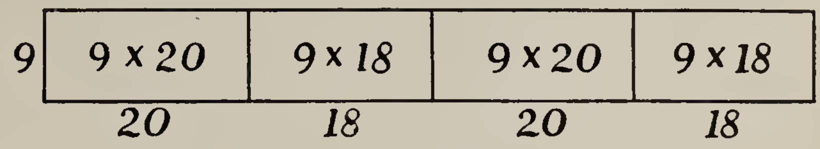


Fig. 9

Since the whole rectangle is equal to the sum of the parts computed above, we have

$$9\times(20+18+20+18)$$
$$=(9\times 20)+(9\times 18)+(9\times 20)+(9\times 18).$$

Simplify both members of this equation and show that they have equal values.

The product $9\times(20+18+20+18)$ may be written briefly without the multiplication sign, as

$$9(20+18+20+18).$$

The products 9×20, 9×18, etc., may be written $9\cdot 20$, $9\cdot 18$, etc.

Exercises

1. Show from a figure that $4(2+1+6)=4\cdot 2+4\cdot 1+4\cdot 6$.

2. Show that $a(b+c)=ab+ac$ (Fig. 10).

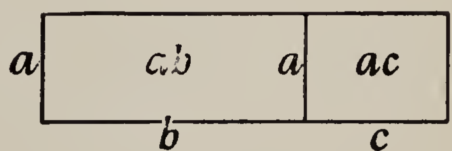


Fig. 10

3. Draw a figure to show that $a(5+2)=5a+2a$.

4. Show from a figure that $m(a+b)=ma+mb$.

5. Show that $a(b+c+d)=ab+ac+ad$, first by means of a figure, second by substituting values for the literal numbers.

11. A rule for multiplying a polynomial by a monomial. Exercises 1 to 5 (§10) illustrate the following important law for multiplying a polynomial by a

monomial: *A polynomial is multiplied by a monomial by first multiplying each term of the polynomial by the monomial, and then adding the resulting products.* This law is to be used in the following exercises:

Exercises

1. Multiply as indicated: $4(x+2)$; $3(x+7)$; $a(s+t)$; $5(y+z+4)$; $3(m+n+1)$; $t(a+b+c+2)$; $(p+q+4)a$; $(f+g+h)m$; $\frac{5(a+c)+2(a+b)}{c}$; $\frac{4(x+y+z)+(a+b)m}{3(x+2)}$.

2. Find the value of each of the parts in Exercise 1 for $a=1$, $b=1$, $c=1$, $f=3$, $g=3$, $h=3$, $m=5$, $n=5$, $p=4$, $q=4$, $r=4$, $s=4$, $t=4$, $x=2$, $y=2$, $z=2$.

Multiply Exercises 3 to 13 as indicated and check as shown in Exercise 3:

3. $7(5a+3)$.

Solution: $7(5a+3)=35a+21$.

Check: Substitute for a some value, first in the exercise, then in the result to see if both reduce to the same number; *e.g.*, let $a=2$.

Exercise		Result
$7(5\cdot 2 + 3)$		$35\cdot 2 + 21$
$7\cdot 13$		$70+21$
91	$=$	91

4. $b(3a+4)$.

5. $2m(a+b)$.

6. $5(2a+3x+2)$.

7. $t(6p+9)$.

8. $\frac{1}{2}b(r+6)$.

9. $3x(4y+2)+5(x+6)$.

10. $\frac{1}{3}m(6a+9)+2(5x+1)$.

11. $p(3m+4)+q(2n+10)$.

12. $2x(4a+2b)+a(3x+5b)$.

13. $3m(2x+8y)+4x(2m+y)$.

14. The base of a rectangle is 2 feet and the altitude is 10 feet. By how much must the base of the rectangle be increased to form a rectangle whose area is 119 square feet?

Suggestion: Let x be the number of feet in the increased part of the base (Fig. 11).

Then the new base is $x+2$.

Hence $10(x+2)=119$

$$10x+20=119$$
$$10x=99$$
$$x=9.9$$

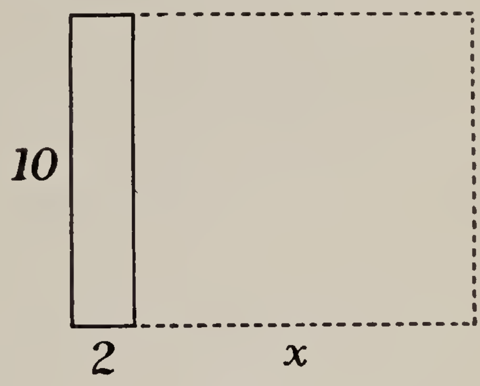


FIG. 11

15. The base of a rectangle is 15 feet and the altitude 1 foot. By how much must the altitude be increased to form a rectangle whose area is 105 square feet?

16. The base of a rectangle is 8 and its altitude is $3x+2$. If the area is 56, find x and the altitude.

Solve the following equations in Exercises 17 to 22 and check each as shown in Exercise 17:

17. $\dfrac{2(x+2)}{3}=8.$

Solution: $\dfrac{\cancel{3}\cdot 2(x+2)}{\cancel{3}}=3\cdot 8$, by multiplying both members by 3

$2x+4=24$, by changing the fraction to the simplest form

$2x=20$, by subtracting 4 from each member

$x=10$, by dividing both members by 2

Check: Substitute 10 for x in the *original* equation and see if both sides of the equation reduce to the same number:

LEFT SIDE		RIGHT SIDE
$\dfrac{2(10+2)}{3}$		8
$\dfrac{2\times \overset{4}{\cancel{12}}}{\cancel{3}}$		8
2×4		8
8	=	8.

18. $3(x+4)=22+x$.

19. $9(y+35)=5(2y+45)$.

20. $3(a+15)+5=2(2a+9)+4(a+3)$.

21. $\dfrac{5(x+3)}{2}=20$. 22. $\dfrac{x+7}{3}=11$.

Multiplication of a Polynomial by a Polynomial

12. Finding the product of two polynomials by means of a rectangle. Divide the rectangle (Fig. 12) into smaller rectangles, as shown in the diagram.

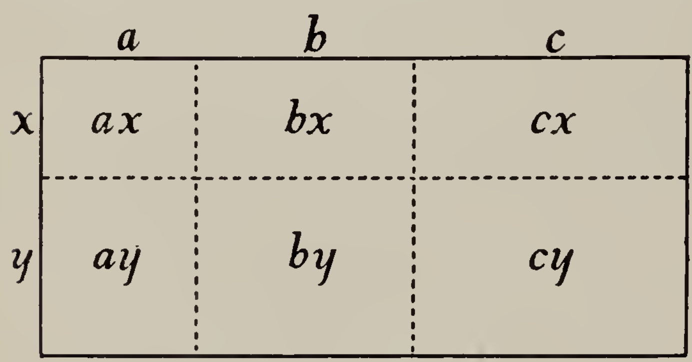


Fig. 12

Then the area of the whole rectangle $(x+y)\ (a+b+c)$ is equal to the sum of the parts

$$ax+ay+bx+by+cx+cy.$$

Hence, $(x+y)\ (a+b+c)=ax+ay+bx+by+cx+cy$.

Exercises

1. Represent the product $(m+n)\,(x+y)$ geometrically as the area of a rectangle, and express it in the form of a polynomial.

2. Express $(a+b)\ (m+n)$ as a polynomial without the use of a drawing.

13. A rule for finding the product of two polynomials. A study of the results in Exercises 1 and 2

above reveals the following law for multiplying polynomials without using a geometric figure:

Two polynomials are multiplied by multiplying each term of one by every term of the other, and then adding the resulting products.

Apply this law to the following products:

Exercises

1. $(a+b)\ (c+d)$.

Solution: $(a+b)\ (c+d)=ac+bc+ad+bd$.

2. $(x+y)\ (m+n+4)$.
3. $(x+2)\ (y+5)$.
4. $(r+8)\ (s+\frac{1}{2})$.
5. $(f+g+6)\ (a+4)$.
6. $(x+2)\ (r+t)$.
7. $(m+n+p)\ (x+y+z)$.

8. Find the values of the polynomials in Exercises 1 to 7, substituting the value 3 for a, b, c and d; 4 for f, g, m and n; $\frac{1}{2}$ for p, r and t; $\frac{1}{3}$ for x and y.

9. Find the value of the polynomial in Exercise 6 when $x=3.52$, $r=1.7$, $t=1.34$.

10. Find the value of the polynomial in Exercise 3 when $x=.18$, $y=3.12$.

Multiply as indicated:

11. $(3x+2y+z)(x+y)$.
12. $(m+3n+4)(2m+3)$.
13. $(2a+8b+3c)(14x+7y)$.
14. $(\frac{1}{3}x+\frac{1}{4}y+\frac{1}{2}z)(\frac{1}{2}m+n)$.
15. $(x+3y+\frac{1}{2}z)\ (a+2b)$.
16. $(a+b)\ (c+d+e)$.
17. $(3m+p+5r)\ (x+y)$.
18. $(\frac{1}{2}f+\frac{1}{3}g)\ (6a+12b+18c)$.
19. $(.5a+.25b)\ (10p+60q+8r)$.
20. $(2x+y)\ (4a+3b+c)$.
21. $(a+b)x+(m+t)+(a+b)$.
22. $4x(m+\frac{1}{3}n)+5y(10a+3b)+16$.
23. $\frac{2}{3}a(3x+y)+\frac{1}{4}b(m+6n)+4$.
24. $a(x+y)+\frac{b}{2}(4m+6t)+8$.

Area of a Square

14. A formula for finding the area of a square. If the base of a rectangle is equal to the altitude the figure is a square (Fig. 13).

Denoting the length of the sides of the square by a, the area is given by the formula

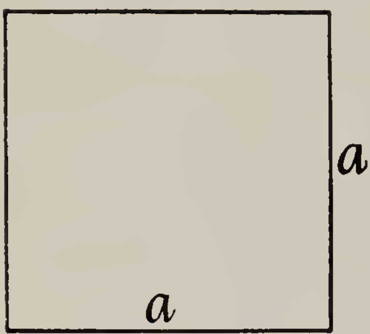


Fig. 13

$$A = a \times a.$$

The product $a \times a$ is written a^2 (read "a-square," or "a to the second power"). In this notation the area of a square is given by the formula

$$\mathbf{A = a^2},$$

where a denotes the length of the side.

The symbol a^2, meaning $a \times a$, should not be confused with $2a$, which means $a + a$, or $2 \times a$.

Exercises

1. Find the area of a square whose side is 1.3.

Solution: $A = a^2$

$a = 1.3$

$A = (1.3)\ (1.3)$, or 1.69

$\therefore A = 1.69.$

2. Find the area of a square whose side is 3.6; 8.4; 1.25.

3. State the meaning of 2^2; $(1.8)^2$; a^2; $(mn)^2$; $(\frac{1}{2})^2$; $(\frac{2}{3})^2$.

4. Tabulate the squares of all whole numbers from 1 to 20, as shown below, and memorize the squares:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
n^2																				

5. State the meaning, and then find the value, of each of the following:

$$(\tfrac{1}{2})^2;\ (\tfrac{2}{3})^2;\ (\tfrac{2}{5})^2;\ (.5)^2;\ (.75)^2;\ 3(\tfrac{2}{3})^2;\ 2(\tfrac{1}{6})^2;\ \frac{1}{2^2\cdot 3^2};\ \frac{3}{4^2\cdot 3^2}.$$

6. The table below contains units of square measure which are commonly used. Verify the first three.

TABLE OF SQUARE MEASURE

144	square inches (sq. in.)	= 1 square foot (sq. ft.)
9	square feet (sq. ft.)	= 1 square yard (sq. yd.)
$30\frac{1}{4}$	square yards (sq. yd.)	= 1 square rod (sq. rd.)
160	square rods (sq. rd.)	= 1 acre (A)
640	acres (A)	= 1 square mile (sq. mi.)

15. How to use rectangles and squares to picture per cents. The square $ABCD$ (Fig. 14) contains 100 small squares. The area of each small square is therefore $\frac{1}{100}$ of $ABCD$, or 1% of $ABCD$.

Similarly 2, 3, 4, 5 squares are 2%, 3%, 4%, 5% of the whole figure.

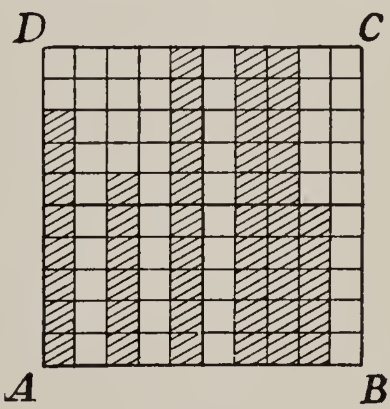


FIG. 14

EXERCISES

1. State the number of per cents in each of the shaded surfaces (Fig. 14).

2. Statistics show that in 1920 our population was divided as follows: 76.7% native whites, 13.0% foreign whites, 10.3% colored. Make a drawing like Fig. 15 to illustrate these facts.

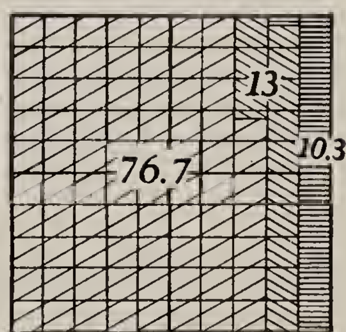


FIG. 15

3. A man divides his farm as follows: he takes 15% for pasture, 50% for corn, 25% for wheat, 7% for meadow, and 3% for lots. Make a drawing like Fig. 15 to show how the farm is divided.

4. The table below states the percentage of illiterates of all people in the United States older than 10 years:

1880	1890	1900	1910	1920
17%	13.3%	10.7%	7.7%	6%

Represent these facts on squared paper as in Fig. 14.

5. In a spelling test containing 100 words Richard spelled 92 words correctly, Mary 95, and John 67. What per cent did each spell correctly? Make a drawing to show these per cents.

16. How to show the relation between important per cents and fractions. By counting the small squares contained in a large square divided into hundredths (Fig. 16), we shall show how to change per cents to common fractions and common fractions into per cents.

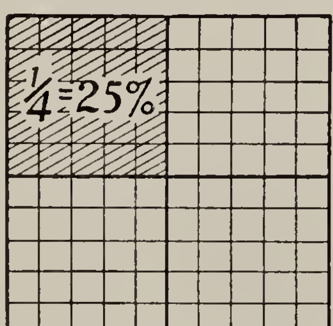


Fig. 16

Exercises

1. Divide a square (Fig. 16) into 100 equal squares and mark off $\frac{1}{4}$ of the large square. Count the number of small squares. What per cent is equal to $\frac{1}{4}$?

What per cent is equal to $\frac{3}{4}$?

2. By means of drawings picture per cents equal to the following fractions: $\frac{1}{2}$, $\frac{1}{8}$, $\frac{3}{8}$, $1\frac{1}{2}$.

3. By means of drawings represent: 6%, $12\frac{1}{2}$%, $33\frac{1}{3}$%.

4. From recent statistics it was learned that the shares of various nations in supplying the world with cotton were as given below. Illustrate the per cents by means of a drawing.

United States	56.7%	Egypt	5.9%	Brazil	2.8%
British India	22.4%	China	5.7%	All others	6.5%

17. How to find the square of a number by means of rectangles and squares. The square of 25 may be found geometrically as follows:

Write 25 as a binomial, as 20+5.

Draw a square whose side is 20+5. (See Fig. 17).

The area of this square is $(20+5)^2$. Why?

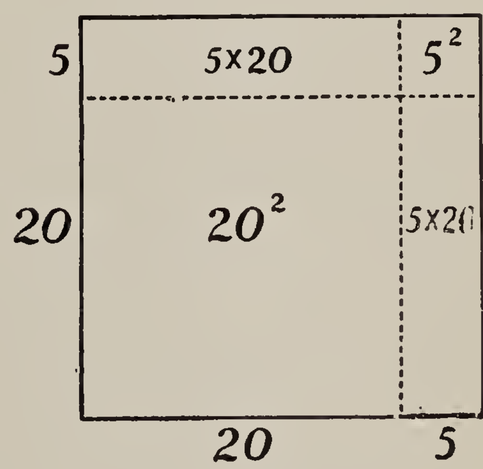


Fig. 17

Divide the square into two rectangles and two squares as shown in the diagram and find the area of each part.

The sum of the four parts is

$$20^2+(5\times20)+(5\times20)+5^2=20^2+2(5\times20)+5^2.$$

It follows that $(20+5)^2=20^2+2(5\times20)+5^2$.

Exercises

By means of drawings find the squares of 13, 26, 34, 29.

18. A rule for squaring a number. We see from Fig. 17 that the square of 25 may be found without a diagram as follows:

1. *Square* 20
2. *Square* 5
3. *Multiply the product* 20×5 *by* 2.
4. *Find the sum:* $20^2+5^2+2(20\times 5)$.

State a rule for squaring a two-figure number.

Exercises

Square the following numbers, doing as much as you can orally. Test each result by multiplying the number by itself directly.

Solution: $62^2=(60+2)^2=60^2+2(60\times 2)+4$

$$\begin{aligned} &= 3600\\ &+\ \ 240\\ &+\qquad 4\\ \hline \therefore 62^2 &= 3844 \end{aligned}$$

2. 24.
3. 35.
4. 41.
5. 74.
6. 38.
7. 82.
8. 105.
9. 109.

19. How to find the square of a binomial by means of a drawing. You may square a binomial $a+b$ as follows:

Draw a segment equal to $a+b$ (Fig. 18).

Draw a square on $(a+b)$ and divide it into four parts as shown.

Show that

$$(a+b)^2=a^2+2ab+b^2.$$

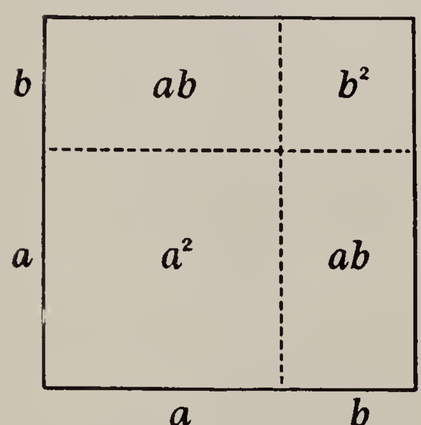


Fig. 18

Exercises

By means of drawings find the squares of the following binomials:

1. $(m+x)^2$.
2. $(c+d)^2$.
3. $(x+2)^2$.
4. $(7+4)^2$.

20. A law for finding the square of a binomial. The formula $(a+b)^2=a^2+2ab+b^2$, translated into words, expresses the following law:

The square of a binomial may be found by squaring the first term, adding twice the product of the two terms, and then adding the square of the second term.

Exercises

Use the formula $(a+b)^2=a^2+2ab+b^2$ to find the trinomials equal to the following squares. Do as many as you can orally. Illustrate some of them with drawings.

1. $(m+n)^2$.
2. $(x+y)^2$.
3. $(x+3)^2$.
4. $(\frac{1}{2}x+\frac{1}{2}y)^2$.
5. $(3a+2b)^2$.
6. $(.4a+.5b)^2$.
7. $(a+\frac{2}{3}b)^2$.
8. $(6x+1)^2$.
9. $(4a+\frac{1}{2}b)^2$.

Find the following products, using the principle of §13:

10. $(2a+3b)\ (a+2b)$.

Solution: $$\begin{aligned}(2a+3b)\ (a+2b)&=2a^2+3ab+4ab+6b^2\\ &=2a^2+7ab+6b^2.\end{aligned}$$

11. $(x+8)\ (x+2)$.
12. $(x+6)\ (x+1)$.
13. $(x+7)\ (x+4)$.
14. $(2z+4)\ (3z+5)$.
15. $(\frac{1}{2}a+\frac{1}{3}b)\ (6a+2b)$.
16. $(\frac{2}{3}x+3z)\ (6x+12z)$.
17. $(5b+2c)\ (5c+3b)$.
18. $(x+2a)\ (3x+b)$.
19. $(a+b+c)\ (x+y)$.
20. $(2x+3y+4)\ (x+y)$.
21. $(a+b+c)^2$.
22. $(2x+y+5)^2$.

21. Trinomial square. The trinomials $m^2+2mn+n^2$, $x^2+2xy+y^2$, x^2+6x+9, are perfect squares. Show that the trinomials, which were found by squaring the binomials in Exercises 1 to 9 (§20) are perfect *trinomial squares.*

Make up three other quadratic trinomials that are squares.

Square Root

22. To find the side of a square whose area is known. A field of the form of a square has an area of 100 square rods. We may find the length of the side of the square as follows:

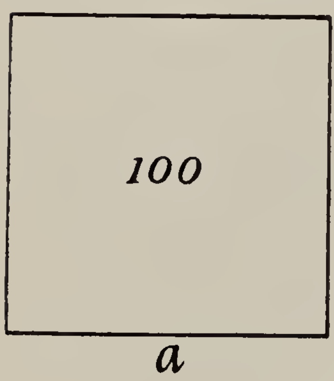


Fig. 19

Let a (Fig. 19) denote the number of rods in one side.

Then $a^2=100$. Why?

This equation may be regarded as expressing numerically the question: What number multiplied by itself gives the product 100?

The answer is 10, since $10\times10=100$.

Hence, $a=10$.

23. Meaning of square root. The process of finding one of the two equal factors whose product is equal to a given number is called *finding the square root of the given number.* Thus 10 is the square root of 100, because $10\times10=100$. The square root *of a number is one of the two equal factors whose product is the number.*

Exercises

1. State the squares of the whole numbers from 1 to 20.

2. Determine the square root of each of the following numbers by finding a number which multiplied by itself gives a product equal to the given number.

64	676	441
169	900	625
400	529	784
484	256	324

24. Radical sign. To indicate that a square root of a number is to be found, the *radical sign* $\sqrt{}$ is used. Thus, "square root of 100" is written briefly $\sqrt{100}$.

Exercises

1. Give the meaning and value of each of the following symbols:

$\sqrt{25}$,

Solution: $\sqrt{25}=5$, since $5\times 5=25$.

$\sqrt{121}$.	$\sqrt{36}$.	$\sqrt{.25}$.
$\sqrt{196}$.	$\sqrt{225}$.	$\sqrt{.81}$.

2. Find the side of a square whose area is 625 square feet.

Solution: Let a denote the number of feet in the side.

$$\text{Then } a^2=625$$

Taking the square root of both members of this equation we have

$$\sqrt{a^2}=\sqrt{625}$$
$$\text{or } a=25.$$

3. Find the side of a square whose area is 144; 81; 324; 256; .25; $\frac{16}{49}$; $\frac{144}{169}$; $\frac{81}{289}$; $\frac{400}{169}$.

25. Finding the square root of a perfect square by factoring. If you know the *squares* of all whole numbers from 1 to 20 you can easily find the *square roots* of all perfect squares from 1 to 400. The process of finding a square root of a perfect square larger than 400 is illustrated in the following example:

Let it be required to find the square root of 11,025.

Solution: Find all the factors (divisors) of 11,025 which contain no other numbers as factors but 1 (prime factors):

$$\begin{array}{r} 5\overline{)11025} \\ 5)2205 \\ 3)441 \\ 3)147 \\ 7)49 \\ 7 \end{array}$$

We may now write

$$\sqrt{11025}=\sqrt{5^2\times3^2\times49}=5\times3\times7$$

Hence, $\sqrt{11025}=105.$

Exercises

Find a square root of each of the following perfect squares. Arrange your work as shown in Exercise 1.

1. 576.

Solution:

$$\begin{aligned}\sqrt{576}&=\sqrt{2\times2\times12\times12}\\&=2\times12\\&=24.\end{aligned}$$

Computation:

$$\begin{array}{r} 2\overline{)576} \\ 2\overline{)288} \\ 12\overline{)144} \\ 12 \end{array}$$

2. 1225.	5. 2916.	8. 3249.
3. 2304.	6. 2601.	9. 3969.
4. 2704.	7. 2401.	10. 4356.

26. Square roots may be found by means of a table. To save time and energy, tables containing numbers and their square roots may be used. In the table on page 28 the columns headed "Nos." contain the integral (whole) numbers from 1 to 104. The columns immediately to the right give the corresponding squares, and the second columns to the right give the square roots of the numbers in the first column. The roots are approximated to four figures.

Exercises

By means of the table on page 28, find the square roots of: 6241; 9216; 3481; 5625.

27. How to tell the number of digits in the square root of a number. In extracting the square root of a number we must know where to place the decimal point. The following shows us how to do this:

State the squares of the *integral* (whole) numbers from 1 to 9, and note that they contain either *one* or *two* digits, and that none contains *more* than two digits. It *follows that the square root of an integral number, which is less than 100 and greater than 1, lies between 1 and 10, and contains only one digit in its integral part.*

Similarly show that the squares of the integral numbers from 10 to 99 contain either three or four digits. It follows that *the square root of an integral number*

TABLE OF SQUARES AND SQUARE ROOTS

Nos.	*Squares*	*Square Roots*	*Nos.*	*Squares*	*Square Roots*	*Nos.*	*Squares*	*Square Roots*
			35	1225	6.916	**70**	4900	8.366
1	1	1.000	36	1296	6.000	71	5041	8.426
2	4	1.414	37	1369	6.082	72	5184	8.485
3	9	1.732	38	1444	6.164	73	5329	8.544
4	16	2.000	39	1521	6.244	74	5476	8.602
5	25	2.236	**40**	1600	6.324	**75**	5625	8.660
6	36	2.449	41	1681	6.403	76	5776	8.717
7	49	2.645	42	1764	6.480	77	5929	8.774
8	64	2.828	43	1849	6.557	78	6084	8.831
9	81	3.000	44	1936	6.633	79	6241	8.888
10	100	3.162	**45**	2025	6.708	**80**	6400	8.944
11	121	3.316	46	2116	6.782	81	6561	9.000
12	144	3.464	47	2209	6.855	82	6724	9.055
13	169	3.605	48	2304	6.928	83	6889	9.110
14	196	3.741	49	2401	7.000	84	7056	9.165
15	225	3.872	**50**	2500	7.071	**85**	7225	9.129
16	256	4.000	51	2601	7.141	86	7396	9.273
17	289	4.123	52	2704	7.211	87	7569	9.327
18	324	4.242	53	2809	7.280	88	7744	9.380
19	361	4.358	54	2916	7.348	89	7921	9.433
20	400	4.472	**55**	3025	7.416	**90**	8100	9.486
21	441	4.582	56	3136	7.483	91	8281	9.539
22	484	4.690	57	3249	7.549	92	8464	9.591
23	529	4.795	58	3364	7.615	93	8649	9.643
24	576	4.898	59	3481	7.681	94	8836	9.695
25	625	5.000	**60**	3600	7.745	**95**	9025	9.746
26	676	5.099	61	3721	7.810	96	9216	9.797
27	729	5.196	62	3844	7.874	97	9409	9.848
28	784	5.291	63	3969	7.937	98	9604	9.899
29	841	5.385	64	4096	8.000	99	9801	9.949
30	900	5.477	**65**	4225	8.062	100	10,000	10.000
31	961	5.567	66	4356	8.124	101	10,201	10.049
32	1024	5.656	67	4489	8.185	102	10,404	10.099
33	1089	5.744	68	4624	8.246	103	10,609	10.148
34	1156	5.830	69	4761	8.306	104	10,816	10.198

containing three or four digits contains only two digits in the integral part.

Likewise *the square root of an integral number of five or six digits contains three digits in the integral part.*

These conclusions are summarized in the following table:

Number of digits in a given number	1 or 2	3 or 4	5 or 6	7 or 8	9 or 10
Number of digits in the square root	1	2	3	4	5

The table suggests a way of finding the number of digits in the integral part of the square root of a given number:

Beginning at the unit digit, mark off periods of two figures each, as 14′44, or 1′36′82′30. The "number of periods" is "the number of digits" in the integral part of the square root. The last period at the left may have only one figure, as in 5′52′25.

In all integral numbers the decimal point is omitted. It is understood to be to the right of the last figure. Hence, periods are marked off from the unit place. When the number has a decimal point, as 36.489, the periods are marked off from the decimal point.

Exercises

By marking off periods, find the number of digits contained in the integral part of the square root of each of the following numbers:

1. 65.	**4.** 84.	**7.** 1.231.	**10.** 3600.
2. 784.	**5.** 676.	**8.** 73493.6.	**11.** 270.4.
3. 9363.6.	**6.** 42875.	**9.** 1849.	**12.** 2916.

28. How to extract the square root of a number. The following example shows how to extract the square root of a given number, and explains the reasons for the steps in the process.

Example: Extract the square root of 676.

1. *Beginning at the decimal point, mark off periods of two figures:* 6′76. This shows that the square root contains two digits in the integral part and tells where to put the decimal point in the square root.

2. *Find the largest square in the first period, 6.*

3. Since this is 4, it follows that *2 is the tens* digit of the required square root, and that the square root lies between 20 and 30.

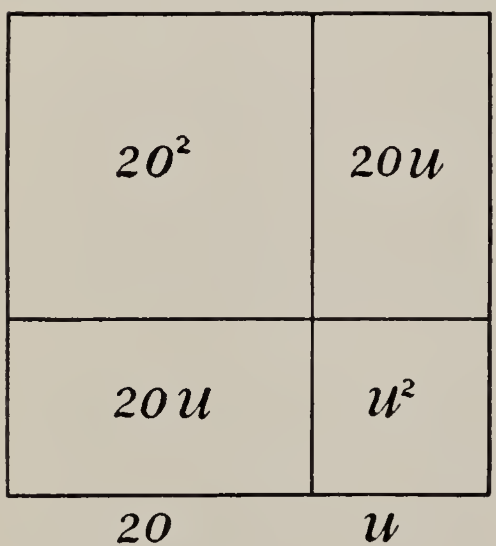


Fig. 20

The unit digit is to be found next. Denoting the unit digit by u, the square root of 676 is equal to $20+u$. Geometrically this means that $20+u$ is the side of a square whose area is 676 (Fig. 20).

Dividing the square into four parts as shown in the diagram, we have

$$676=(20+u)^2=20^2+2\times 20u+u^2,$$
$$\text{or } 676=20^2+2\times 20u+u^2.$$

4. *Subtracting* 20^2 from both sides, we have the remainders

$$676-20^2=2\times 20u+u^2,$$
$$\text{or } 276=40u+u^2.$$

5. The value of u *may now be found by trial* as follows: Disregarding for the moment the u^2 and dividing by 40, *i.e.*, dividing 40 into 276, or 4 into 27, we find the quotient to be almost 6.

6. By trying u equal to 6, we have

$$40u+u^2=(40\times 6)+6^2=240+36=276$$

It follows that $u=6$.

Hence, $$676=(20+u)^2=(20+6)^2$$

and $$\sqrt{676}=20+6=26.$$

The preceding work may now be arranged briefly in the following convenient form:

1. *Write down the number 676 whose square root is to be found.*

2. *Mark off periods and find the largest square contained in the first period.* This gives 4.

3. *Write the square root of 4 over the first period.*

$$\begin{array}{r|r} & 2\ \ 6 \\ \hline & 6'76 \\ & 4 \\ \hline 46 & 2\ 76 \\ & 2\ 76 \\ \hline \end{array}$$

4. *Write the 4 under the first period and subtract 4 from 6. Bring down the second period 76.*

5. *As a trial divisor use twice the part of the root thus far found, i.e., 2×2, and divide this into the remainder 276 leaving off the last digit 6 for convenience. This gives 6. Annex the 6 just found to the trial divisor 4 and to the part of the root so far found.*

6. *Multiply the new divisor, 46, by 6 and write the result, 276, under the remainder.*

If the given number has more than two periods, repeat steps 4 to 6 until all periods are used.[1]

[1] A brief method for finding the square root of a number approximately is given in §169.

Exercises

Extract the square root of each of the following numbers, arranging the work as in Exercise 1, below:

1. 73.61′64.

Solution:

$$\begin{array}{r|l} & 8.\ 5\ \ 8 \\ \hline & 73.'61'64 \\ & 64 \\ \hline 165 & 9\ 61 \\ & 8\ 25 \\ \hline 1708 & 1\ 36\ 64 \\ & 1\ 36\ 64 \\ \hline \end{array}$$

$$\therefore \sqrt{73.6164} = 8.58.$$

2. 9801.
3. 2079.36.
4. 82.81.
5. 6084.
6. 43.56.
7. 2916.
8. 3025.
9. 1.5129.
10. 20736.
11. 9.3636.
12. 462.25.
13. 22.5625.

Extract the square root of each of the following numbers and carry the process to two decimal places.

14. 5.
15. 8.
16. 18.
17. 1.24.
18. 25.64.
19. 3.243.
20. 421.6.
21. 50.43.
22. 65.48.
23. 739.4.
24. 8.364.
25. 98.36.

26. It is found by experiments that the approximate distance d passed over in the time t by an object falling from rest, is given by the formula $d = 16t^2$. State this formula in words.

Find the distance an object falls in 3 seconds; 10 seconds; 18 seconds.

In how many seconds does an object drop 576 feet? 1296 feet? 176 feet?

29. The theorem of Pythagoras. Draw a right triangle (Fig. 21) whose sides are three, four, and five units long.

Draw a square on each side.

Divide the square on the hypotenuse into parts as shown in the diagram.

Show that I+II $=3\times4=12$.

Since I = II it follows that I = 6 and II = 6.

Similarly III = IV = V = 6.

Therefore the area of the square on the hypotenuse c is II + III + IV + V + 1 $=4\times6+1=25$.

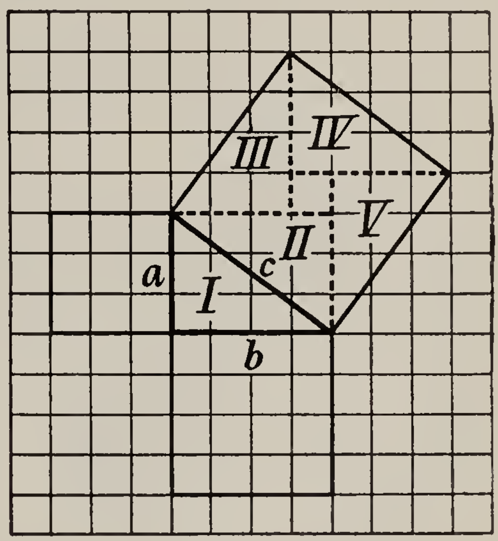


FIG. 21

$$\text{Briefly } c^2=25$$
$$\text{But } a^2=3\times3=9$$
$$\text{and } b^2=4\times4=16$$
$$\text{Hence } a^2+b^2=25$$
$$\text{It follows that } a^2+b^2=c^2.$$

This formula is a very important relation. It holds for the three sides of any *right* triangle. If two sides of a right triangle are known it enables us to determine the third side. Translated into words it may be stated as follows:

The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.

This principle is known as the *Theorem of Pythagoras*, being named after the Greek mathematician, Pythagoras. It is one of the famous theorems of geometry. Although it was known before the time

of Pythagoras, he is believed to have been the first to give a general proof. The story is told that Pythagoras, jubilant over his great accomplishment of proving this valuable theorem, made a sacrifice to the muses who inspired him.

PYTHAGORAS

Pythagoras was born at Samos about 569 B.C., of Phœnecian parents, and died, probably at Metapontum, in southern Italy, about 500 B.C. He was primarily a moral reformer and philosopher, but he was also a geometer, an arithmetician, and a teacher of astronomy, mechanics, and music. He is said to have been the first to employ the word mathematics. The meaning he gave it was what we understand by general science. With him geometry meant about what people to-day mean by mathematics.

Exercises

Solve the following problems using the theorem of Pythagoras. In each case make a diagram.

1. The main pole of a tent is 36 feet high. In putting up the tent, guy wires are to be attached to the top, and to stakes 48 feet from the foot of the pole. How long must they be, excluding the length used for tying?

Solution: The pole, the wire, and the line from the foot of the pole to one of the stakes form a *right* triangle.

Make a diagram (Fig. 22).

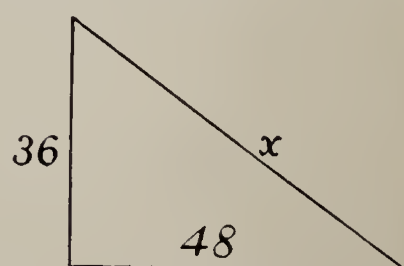


Fig. 22

By the theorem of Pythagoras, we have

$$x^2 = 48^2 + 36^2$$
$$x^2 = 2304 + 1296$$
$$x^2 = 3600.$$

Extracting the square root of both sides of the equation, we have

$$x = 60.$$

2. Find the length of a guy wire used to brace a telephone pole 30 feet high, if it is to be attached to a stake 12 feet from the foot of the pole.

3. A 20-foot ladder leans against a wall. If the foot of the ladder is 6 feet from the base of the wall, how high above the ground is the top of the ladder?

Solution: Let x be the number of feet required.

Then $x^2 + 6^2 = 20^2$

$$x^2 + 36 = 400$$

Subtracting 36 from both sides of the equation, we have

$$x^2 = 364$$

Extracting the square root of both members, we have

$$x = \sqrt{364} = 19 \text{ approximately.}$$

4. A surveyor has to find the third side of a right triangle whose hypotenuse is 20 rods, and one of whose sides is 12 rods. How long is the remaining side?

5. The diagonal distance joining two opposite vertices of a square is 18. Find the length of the side.

6. Find the length of the diagonal of a rectangle 36 feet × 64 feet.

7. If the side of a baseball diamond is 90 feet, how far is it directly from *home* to *second?*

8. Measure the sides of a sheet of notebook paper. Find, without measuring, the distance between two opposite corners. Check your result by measuring the distance.

9. To find the distance, AB, across a swamp (Fig. 23) a right triangle ABC is laid off.

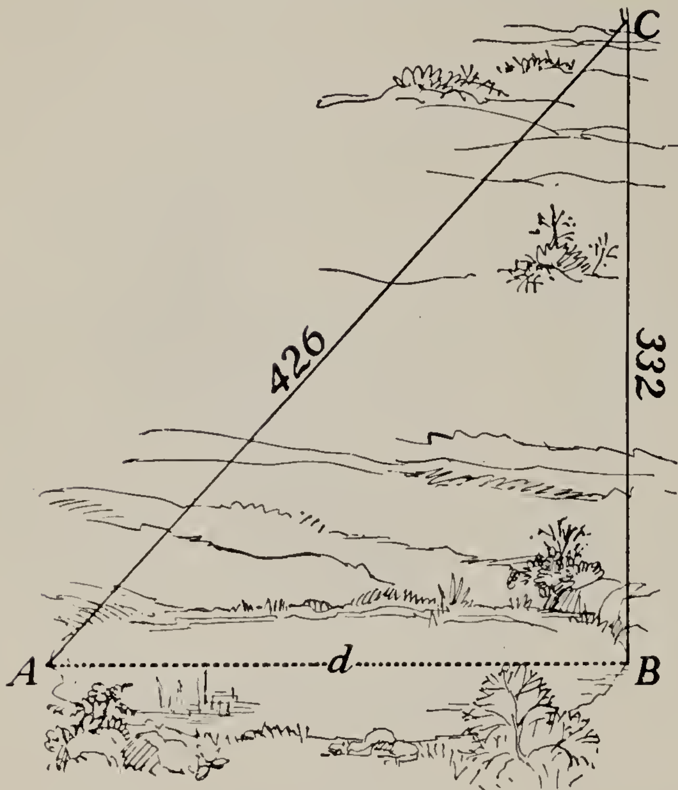


FIG. 23

Side BC is measured and found to be 332 feet long.

AC is found to be 426 feet long. Find AB.

Solve the following equations as shown in Exercises 1 and 3:

10. $x^2+8^2=10^2$.

11. $35^2-21^2=x^2$.

12. $x^2+729=2025$.

13. $144+x^2=225$.

14. $25^2-x^2=20^2$.

15. $1600-x^2=576$.

16. $900-x^2=324$.

17. $400-x^2=256$.

30. What every pupil should know and be able to do. The study of Chapter I should enable every pupil to do the following:

1. To solve problems about areas of rectangles and squares by means of the formulas $A=ab$, $A=a^2$.

2. To find the length of one side of a right triangle when the lengths of the other two sides are known.

3. To extract the square root of a given number.

4. To find the value of a polynomial, as

$$ab+cd+ef+g^2$$

for given values of the literal numbers.

5. To solve equations like $3(x+4)=22+x$; $\frac{2(x+2)}{3}=8$; $a^2=169$; $x^2+36=400$.

6. To make the graph of an equation like $A=5h$.

7. To multiply a polynomial by a monomial, and to multiply a polynomial by a polynomial.

8. To find the product when one factor is zero.

9. To state the squares of whole numbers from 1 to 20.

The following facts and principles should be known:

10. The formulas for finding the area of a rectangle and of a square.

11. The law of order in multiplication.

12. The theorem of Pythagoras.

13. $(a+b)^2=a^2+2ab+b^2$.

31. Typical problems and exercises. The problems and exercises below review the essentials of Chapter I. You should be able to work them correctly.

1. Draw a rectangle 2.3 centimeters $\times$ 1.6 centimeters, and find the area by formula.

2. Make the graph of the equation $A=5h$.

3. State the law of order in addition; in multiplication.

4. The base of a rectangle is 2 feet and the altitude 10 feet. By how much must the base be increased to form a rectangle whose area is 110 square feet?

5. Find the length of the diagonal of a rectangle 36 feet $\times$ 64 feet.

6. To the top of a pole 18 feet high, a 25-foot rope is attached. How far will the rope reach from the foot of the pole?

Multiply as indicated:

7. $8\frac{3}{4}\times 14\frac{2}{3}$.

8. $.06\times .36$.

9. $a(x+y+z)$.

10. $(a+b+c)\ (m+n)$.

11. $(m+n)^2$.

12. $(3a+\frac{1}{2}b)^2$.

13. $(2x+.3y)^2$.

14. $(3r+\frac{1}{6}x)^2$.

15. If $a=12$, $b=3.1$, $c=2$, $d=4.2$, find the value of $a\times 0$; $0\times c$; $\frac{bd+ba}{ad-bc}$; $2ab+cd+3ac+bd$.

Solve each of equations in Exercises 16 to 20 and check:

16. $3(x+4)=22+x$.

17. $\frac{2(a+2)}{3}=8$.

18. $a^2=625$.

19. $x^2+729=2025$.

20. $1600-x^2=576$.

21. Find the square root of 3969; 2916; 98.36.

22. Write a paper on one of the following topics:

a. Uses of the formula for finding the area of a rectangle or square.

b. The use of the rectangle in multiplying algebraic, or arithmetical, numbers.

c. The life and work of Pythagoras, and his famous theorem.

CHAPTER II

AREAS OF QUADRILATERALS, TRIANGLES, AND CIRCLES

The Area of a Parallelogram

32. What we shall study in this chapter. In Chapter I we have learned how to find the area of a rectangle and square. We shall now take up the problem of finding areas of other well-known figures. Notice that the city blocks of a business section (Fig. 24) are triangles,

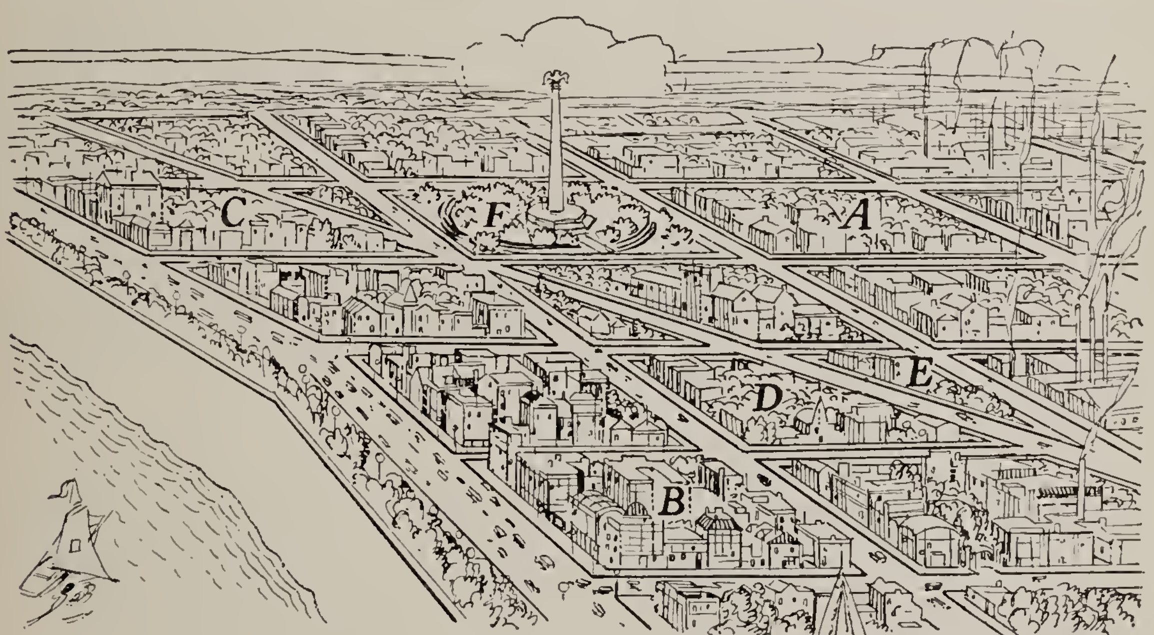


Fig. 24

rectangles, squares, or other quadrilaterals which are not rectangles or squares. The regularity of the occurrence of such figures may be broken by circular surfaces, such as circular fountains or circular parks.

Thus, block A is a rectangle, B is a square, C is a parallelogram, D is a trapezoid, E is a triangle, and F is a circular park. We shall work out the formulas for finding the areas of these surfaces. The formulas will be used to solve problems.

FIG. 25

33. Meaning of parallelogram. A quadrilateral (Fig. 25) whose opposite sides are parallel is a **parallelogram.**

34. How to construct a parallelogram. Draw two intersecting lines, as AB (Fig. 26) and AD.

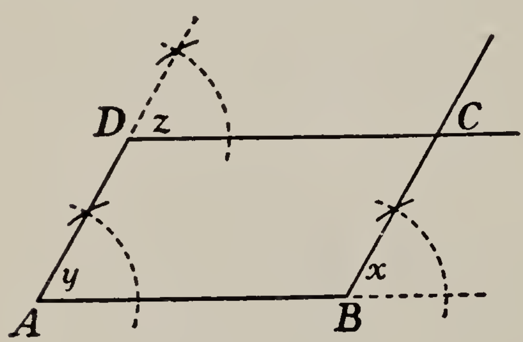


FIG. 26

Following the suggestion of the drawing construct at point B angle x equal to angle y, using only compass and straightedge. Check your construction by measuring the angles with a protractor.

At D construct $z = y$ and check the construction with the protractor.

$ABCD$ is the required parallelogram.

35. Some important properties of the parallelogram. The following exercises establish two important properties of the parallelogram:

EXERCISES

1. Measure each side of the parallelogram $ABCD$ (Fig. 26). How do the opposite sides AB and DC compare as to length? Compare AD and BC.

The results show that *the opposite sides of a parallelogram are equal.*

2. Show that *a diagonal divides a parallelogram into two congruent triangles.*

Suggestion: Draw a parallelogram as shown in §34. Draw a diagonal. Cut out the two triangles into which the parallelogram is divided and show that they can be made to coincide.

3. Measure the angles of a parallelogram and show that *the opposite angles of a parallelogram are equal.*

4. The fact that the opposite sides of a parallelogram are equal suggests the following simple construction of a parallelogram:

Draw AB and AD (Fig. 27).

With radius equal to AD, and with B as center, draw an arc near C.

With radius equal to AB and with D as center, draw an arc cutting the first arc.

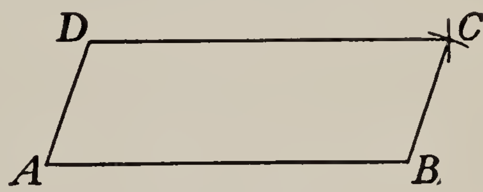


FIG. 27

Let C be the intersection of the two arcs.

Draw BC and DC.

Then $ABCD$ is the required parallelogram.

5. Construct the notebook cover design shown in Fig. 28.

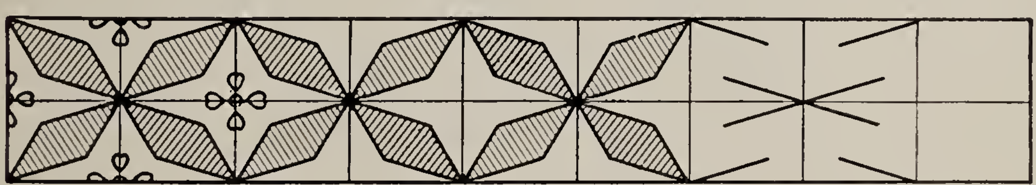


FIG. 28

6. Construct the design shown in Fig. 29.

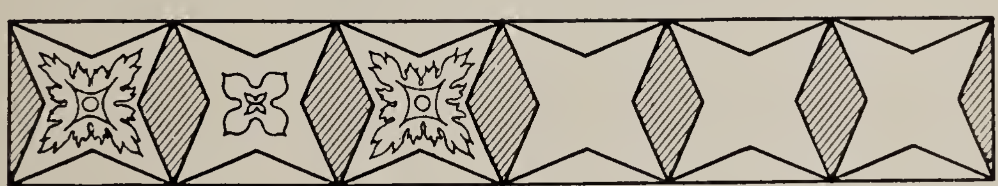


FIG. 29

36. How to find the area of a parallelogram. To find the area of a parallelogram we may proceed as we did with the rectangle, *i.e.*, we may place it upon

squared paper and count the number of square units contained in it (Fig. 30). In most cases this process

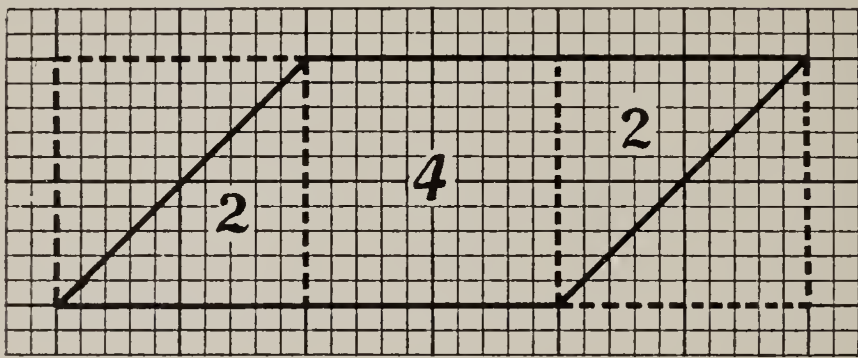


Fig. 30

would be neither so convenient nor so practical as that of finding the area by means of a formula. A formula can be easily worked out for any parallelogram as follows:

Draw the parallelogram $ABCD$ (Fig. 31).

From B draw BC' perpendicular to the side DC.

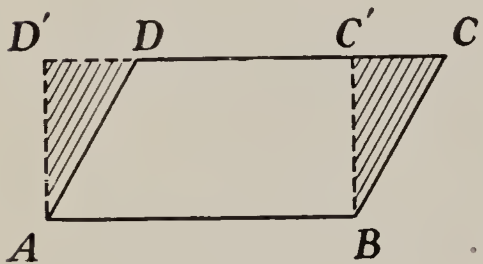


Fig. 31

This cuts off a triangle BCC'.

Move this triangle to the position ADD'.

Then rectangle $ABC'D'$ has the same area as parallelogram $ABCD$.

Since the area of the rectangle is $(AB)\cdot(BC')$, the area of the parallelogram is also $(AB)\cdot(BC')$.

Calling AB the *base* of the parallelogram, and BC' the height, or *altitude*, we may state the result in the form of the following theorem:

The area of a parallelogram is equal to the product of the base by the altitude. Translating this statement into symbols, we have the formula

$$\mathbf{A = bh.}$$

Notice that this formula is similar to that for finding the area of a rectangle. It should be clear, however, that in the parallelogram the altitude is *not* the same as the side.

Exercises

1. Measure the dimensions of blocks A, B, and C (Fig. 24) and find the area of each, using the scale 1 inch equals 600 feet.

2. Find the area of $ABCD$ (Fig. 31).

3. On squared paper draw a parallelogram whose base is equal to 6 centimeters, whose altitude is equal to 2 centimeters, and in which the side adjacent to the base is equal to 2.8 centimeters. Find the area by means of the formula. Check your result by counting unit squares.

4. Find the area of a parallelogram whose base is 12.4 feet and whose altitude is $4\frac{2}{3}$ feet. Use the formula and approximate your result to the nearest third figure. Then consider 12.4 and $4\frac{2}{3}$ as *exact*, and find the exact area.

5. By means of an equation find the altitude of a parallelogram whose area is 26 square inches and whose base is 2 inches.

6. The altitude of a parallelogram is 3 inches and the base 6 inches. By how much must the base be increased to form a parallelogram whose area is 84 square inches?

Suggestion: Make a sketch showing the 3, 6, 84, and the unknown increase.

Solve the problem by means of an equation.

7. The base of a parallelogram is denoted by $4x+7$, the altitude is 5, and the area is 75. Draw a sketch. Find x and the base.

8. Let the altitude of a parallelogram vary (change), the base remaining the same. Let the base be 4.6, and arrange in the form of a table (§7) the values of the area for the following values of the altitude: 1, 1.3, 1.6, 2, 2.8, 3.7, 3.9, 4.

From the table find in each case the ratio of area to altitude and show that the area *varies directly* as the altitude (§6).

Make the graph of the equation $A = (4.6)h$.

THE AREA OF A TRAPEZOID

37. A formula for finding the area of a trapezoid. Draw a quadrilateral with one pair of opposite sides parallel, as $ABCD$ (Fig. 32).

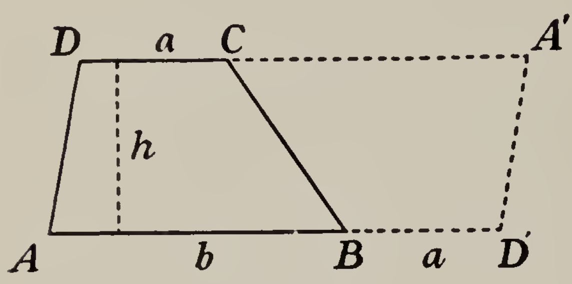


FIG. 32

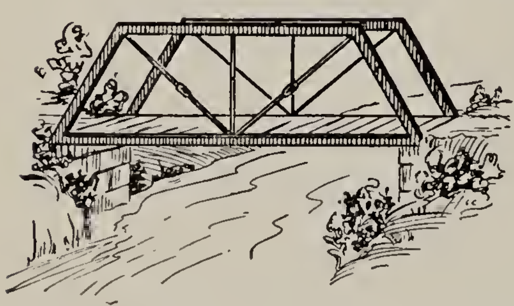
A quadrilateral having two, and only two, of its sides parallel to each other is called a **trapezoid.** The trapezoid is frequently used in the construction of bridges and buildings.

The parallel sides AB and DC are the *bases* of the trapezoid, and the perpendicular distance h between the parallel sides is the *altitude.*

Using the formula for the area of a parallelogram we shall now work out a formula for the area of the trapezoid as follows:

Place a trapezoid of the same size and shape as $ABCD$ in the position $BD'A'C$.

The two trapezoids together form the parallelogram $AD'A'D$ whose base is $a+b$ and whose altitude is h.



$\therefore$ The area of $AD'A'D$ is $h(a+b)$ (§36).

Hence the area of the trapezoid is $\frac{1}{2}h(a+b)$, *i.e., the area of a trapezoid is equal to one-half of the altitude multiplied by the sum of the bases.*

Denoting the area by A, we have the formula

$$\mathbf{A = \tfrac{1}{2}h(a+b).}$$

EXERCISES

1. The bases of a trapezoid are 7 inches and 5 inches and the altitude is 3 inches. Find the area.

Solution: $A=\frac{1}{2}h(a+b)$. Why?

$a=7$, $b=5$, and $h=3$.

$\therefore A=\frac{1}{2}(3)(7+5)=18$

Hence, Area = 18 square inches.

2. Measure the bases and altitude of the trapezoid (Fig. 32) and find the area.

3. On squared paper draw a trapezoid. Find the area by means of the formula. Check your result by counting squares.

4. Make a drawing of the floor plan shown in Fig. 33, using the scale 1 inch equal to 24 feet.

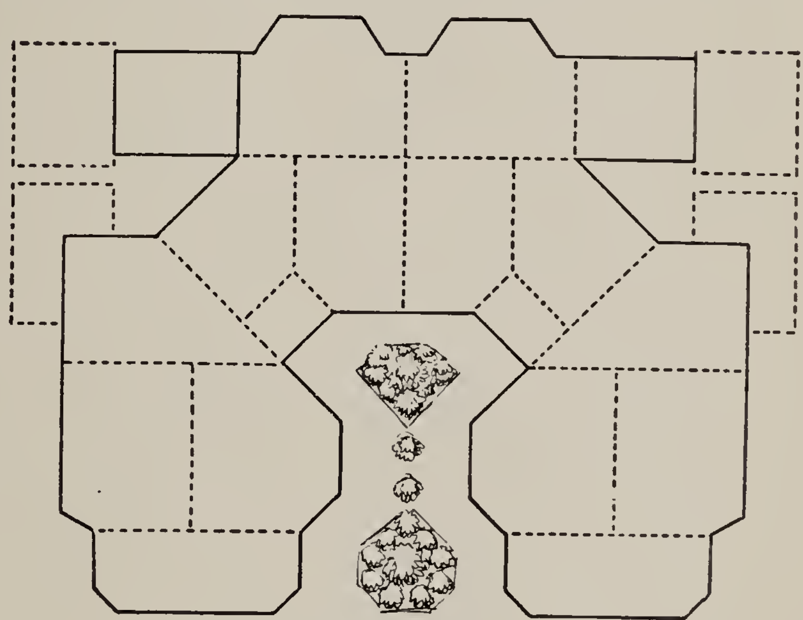


FIG. 33

Divide the plan into rectangles and trapezoids and find the area.

5. Show by letting $a=2$, $b=3$, $h=4$, that

$$\frac{1}{2}h(a+b)=\frac{h}{2}(a+b)=h\frac{(a+b)}{2}=\frac{h(a+b)}{2}$$

Using the method shown in Exercise 1, find the areas of trapezoids whose bases and altitudes are given in the table below.

	Base a	*Base b*	*Altitude h*	*Area A*
6.	9 ft.	4 ft.	6 ft.	
7.	$10\frac{1}{2}$ in.	$6\frac{1}{2}$ in.	7 in.	
8.	13.5 cm.	5.4 cm.	4 cm.	

9. Find the area of block D (Fig. 24), using the scale 1 inch equal to 600 feet.

10. The area of a trapezoid is 324 square inches, and the bases are 27 inches and 9 inches. Make a sketch and find the height by means of an equation.

11. One of the bases of a trapezoid is 7 inches, the area is 45 square inches, and the altitude is 5 inches. Find the length of the other base.

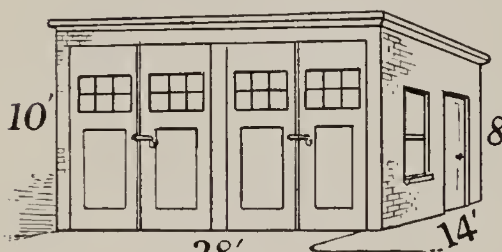


FIG. 34

12. Find the number of square feet of siding needed for the garage (Fig. 34), making no deductions for windows and door.

13. Find the cost of painting the walls of the garage (Fig. 34) at the rate of 50 cents per square yard, making no deductions for windows and door.

14. The trapezoid (Fig. 35) represents the cross section of a trench. Find the area.

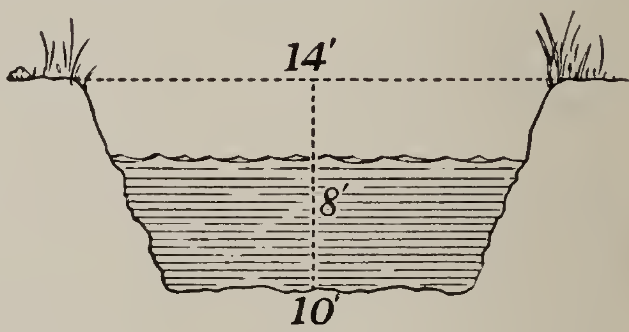


FIG. 35

Solve the following equations and check the results:

15. $\dfrac{7(x+6)}{2}=63.$

Solution:

$$\frac{7(x+6)}{2}=63$$

$$\frac{\cancel{2}\cdot 7(x+6)}{\cancel{2}}=63\cdot 2$$

$$7x+42=126.$$

$$7x=84.$$

$$x=12.$$

Check:

LEFT SIDE		RIGHT SIDE
$\dfrac{7(12+6)}{2}$		63
$\dfrac{7\times \overset{9}{\cancel{18}}}{\cancel{2}}$		63
63	=	63

16. $\dfrac{9(3x+5)}{2}=117.$ **17.** $\dfrac{2(5x+3)}{3}=22.$ **18.** $\dfrac{4(3x+4)}{5}=32.$

19. It has been found that the normal weight of an adult taller than 5 feet may be computed approximately from the formula $\frac{11(d+20)}{2}=w$, where d is the number of inches of the height above 5 feet, and w the weight in pounds.

Find the normal weights of adults of the following heights: 5 feet 1 inch; 5 feet 2 inches; 5 feet 3 inches; 5 feet 4 inches; 5 feet 10 inches.

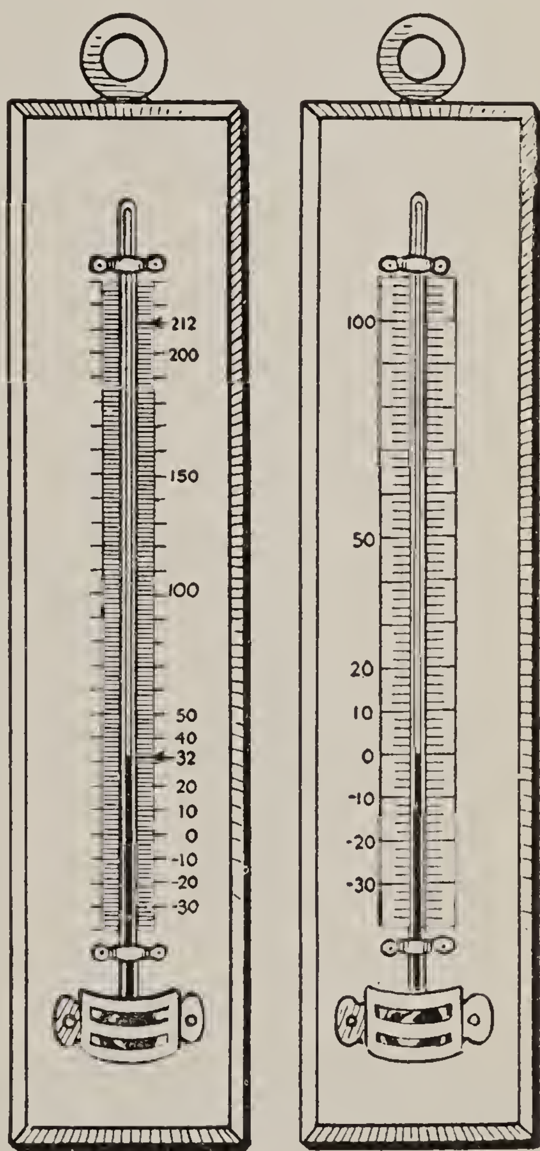
Make a table of the corresponding values of d and w and draw a graph representing the equation of normal weight $\frac{11(d+20)}{2}=w$.

From this graph find the normal weight of a man whose height is 5 feet $4\frac{1}{2}$ inches; 5 feet 3.4 inches; 5 feet $5\frac{2}{3}$ inches.

20. The *Fahrenheit* thermometer (Fig. 36) is the standard instrument in the United States for measuring temperature. On it the freezing point of water is at the 32 degree mark above zero, and the boiling point at 212 degrees above zero.

In *scientific* work the *Centigrade* thermometer is used. On it the freezing point is marked 0 degrees, and the boiling point 100 degrees.

The number of degrees of Fahrenheit may be found with a Centigrade thermometer, using the formula $F=\frac{9C+160}{5}$, where C is the number of degrees Centigrade and F the number of degrees Fahrenheit.



Fahrenheit Thermometer

Centigrade Thermometer

FIG. 36

a. By means of the formula, find the number of degrees Fahrenheit when the Centigrade thermometer reads 0°; 5°; 10°.

b. Make a graph of the equation $F=\frac{9C+160}{5}$. (See §7.)

c. From the graph determine, as nearly as you can, the number of degrees Fahrenheit when the Centigrade thermometer reads $3\frac{1}{2}$°; 8.6°; 5.7°.

The Area of a Triangle

38. Need for a formula for finding the area of a triangle. We have been studying formulas for finding the areas of pieces of land of the shape of rectangles, squares, parallelograms, and trapezoids. We shall also learn how to find the area of an irregular-shaped surface (Fig. 37).

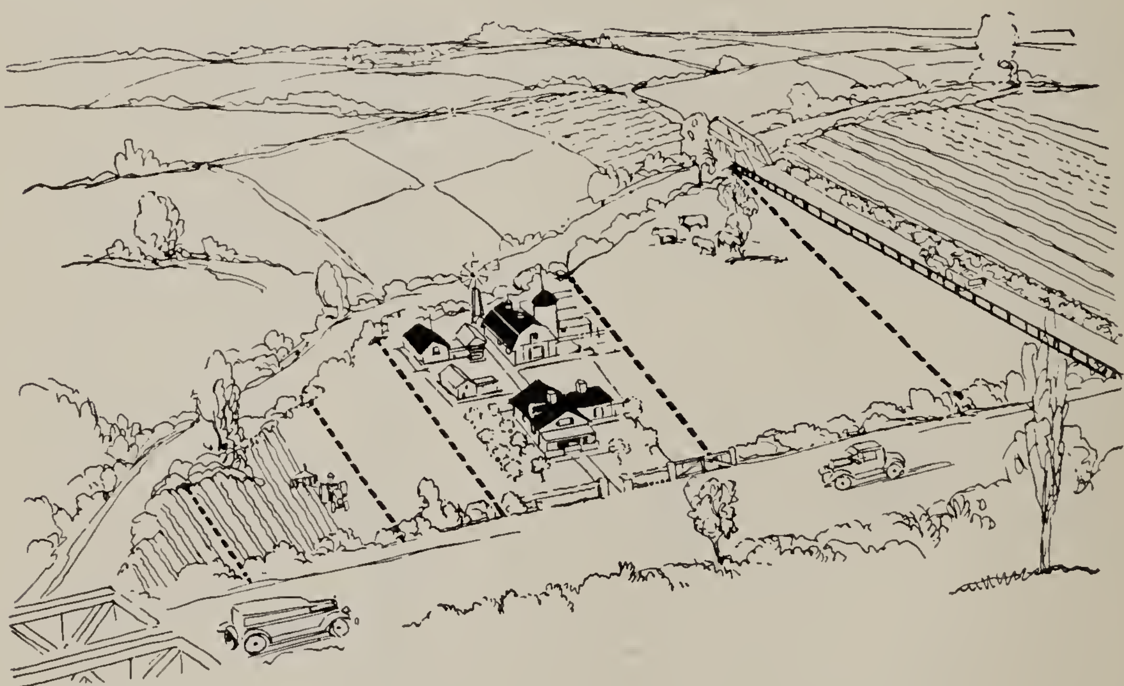


Fig. 37

The area may be found by dividing the surface to be measured into parts having shapes of trapezoids and triangles. Hence, we see the need for a formula for finding the area of a triangle. The parts of the surface

are measured separately, and the area of each is found. The sum is the required area, since the whole is equal to the sum of the parts.

39. How to find the area of a triangle. We may use two methods to find the area of a triangle.

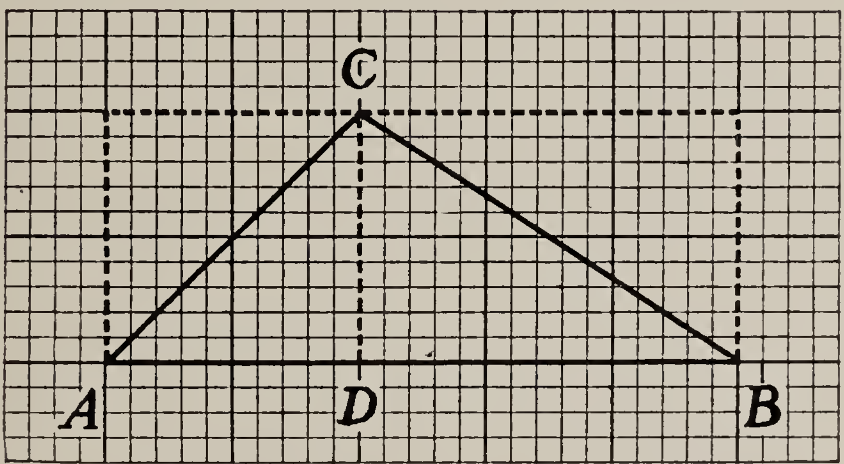


FIG. 38

1. Place the triangle ABC on squared paper (Fig. 38). Note that each of the triangles ADC and BDC is one-half of a rectangle.

Find the number of unit squares in triangles ADC and DBC and add them.

The result is the required area.

2. We may develop a *formula* for finding the area of the triangle. This is done as follows:

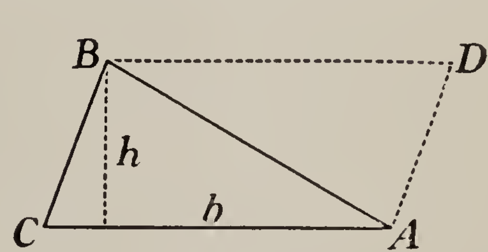


FIG. 39

Through the vertex B of triangle ABC (Fig. 39), draw line $BD \parallel CA$.

Through A draw line $AD \parallel CB$.

The quadrilateral $ADBC$ is a parallelogram. Why?

Since AB divides this parallelogram into two congruent triangles (§35), the area of $\triangle ABC = \frac{1}{2}(ADBC)$.

$$\text{But } ADBC = b \cdot h. \text{ (See §36.)}$$
$$\therefore \text{The area of the triangle} = \tfrac{1}{2} b \cdot h,$$

i.e., The area of a triangle is equal to one-half of the product of the base by the altitude.

Stating this fact in symbols, we have the following formula

$$\mathbf{A = \tfrac{1}{2}bh.}$$

Exercises

1. Measure the base and altitude of $\triangle ABC$ (Fig. 39). Find the area.

2. Show that $\frac{1}{2}b \cdot h = \frac{b \cdot h}{2} = \frac{b}{2} \cdot h = b \cdot \frac{h}{2}$, by substituting values for b and h.

Fig. 40

3. The base line of a triangular park is 34 rods, and the altitude is 18 rods. Find the area in square rods.

4. The span of a roof (Fig. 40) is 24 feet and the height is 7 feet. How many square feet of siding will be needed for the gable?

Find the areas of triangles with the bases and altitudes given in Exercises 5 to 8 below:

	Bases	*Altitudes*
5.	12 ft.	7 ft.
6.	20.6 ft.	18 ft.
7.	$32\frac{1}{2}$ ft.	17.6 ft.
8.	2 ft. 8 in.	1 ft. 6 in.

Find the bases of triangles having the areas and altitudes as shown in Exercises 9 to 13:

9. Area = 48 square inches; altitude = 16 inches.

Solution: $A = \frac{1}{2} b \cdot h$

Hence $48 = \frac{1}{2} b \times 16$

and $48 = 8b$

By dividing both members of this equation by 8, we have the result

$6 = b.$

	Areas	*Altitudes*
10.	28 sq. ft.	5 ft.
11.	30.5 sq. ft.	6 ft.
12.	70.4 sq. ft.	10.4 ft.
13.	125 sq. ft.	18 ft.

14. The base of a triangle is 6. If the base remains the same and if the altitude varies, what formula expresses the area of all of the resulting triangles? Show that the area *varies directly* as the altitude (§6).

15. Make a graph of the equation $A = 3h$.

Suggestions: Complete the table (Fig. 41).

h	A
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

FIG. 41

Plot the pairs of corresponding numbers in the table.

Draw the graph.

From the graph find A, when $h = 4\frac{1}{2}$, $3\frac{1}{2}$, $2\frac{1}{2}$, $\frac{1}{2}$, 0.

16. Draw an obtuse triangle and find the area as in Exercise 1.

17. Find the area of polygon *ABCDEF* (Fig. 42).

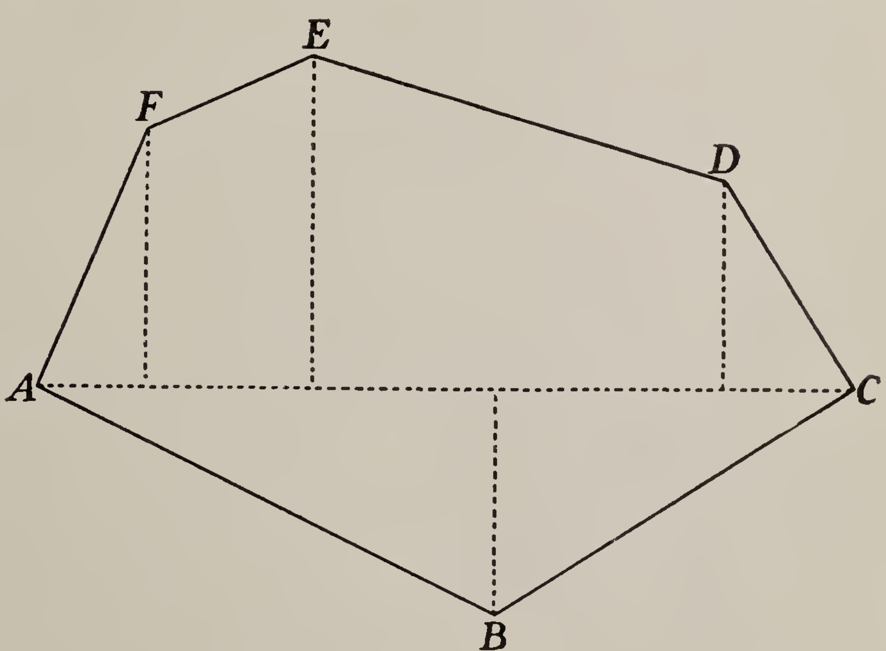


FIG. 42

40. How to represent equations graphically. We have seen repeatedly that the process of making graphs of equations containing two variables, *e.g.*, $c=3.14d$, involves the following steps:

1. Writing the equation.
2. Tabulating pairs of corresponding values of the two variable literal numbers.
3. Drawing the two reference axes.
4. Selecting convenient units for laying off the numbers in the table.
5. Plotting the pairs of corresponding numbers given in the table.
6. Drawing the graph.

Exercises

Make a graph of each of the following equations, using the same axes for all. Follow the directions given above.

1. $y=4x$. **2.** $y=2x$. **3.** $y=6x$.

The Area of the Circle

41. How to find the area of a circle. Draw a circle whose radius is 1.5 inches long.

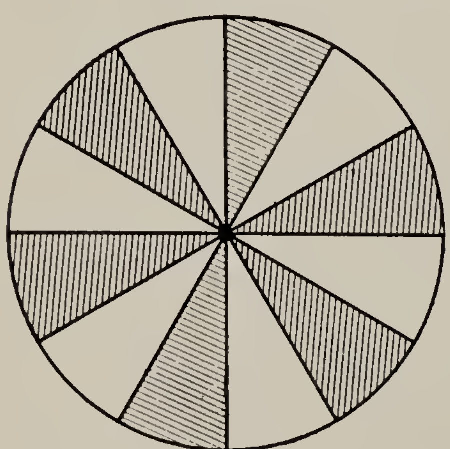


Fig. 43

With the compass or protractor, divide the circle into a number of equal parts, *e.g.*, twelve equal parts (Fig. 43).

Draw the radii to the points of division.

This divides the interior of the circle into twelve congruent surfaces, each bounded by

an arc and two radii, each forming a figure called a *sector*.

The measure of the interior of a sector is called the *area* of the sector.

The sum of the areas of the sectors is the *area* of the circle.

Note that the sectors of the circle are nearly triangular in shape, the base being an arc of the circle and the altitude the radius.

Cut the sectors from the circle and arrange them as in Fig. 44. Figure $AGHK$ thus formed is approximately a parallelogram, the base being equal to one-half of the circumference of the circle, and the altitude being equal to the radius.

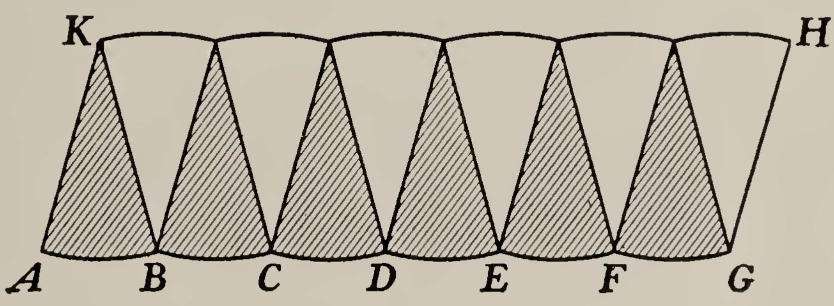


FIG. 44

The area of $AGHK$, which is also the area of the circle, therefore seems to be approximately equal to one-half of the circumference multiplied by the radius. Stated in symbols, this gives the formula

$$\mathbf{A = \tfrac{1}{2}cr.}$$

It can be shown by more advanced methods of mathematics that this formula gives the exact area of a circle.

We know that $C = \pi d = 2\pi r$.

Substituting $2\pi r$ in place of c in the formula above, it follows that the area

$$A = \tfrac{1}{2}(2\pi r)(r) = \pi r^2$$

Hence, $\mathbf{A = \pi r^2}$.

This formula expresses the area of a circle in terms of the radius.

Comparing the circumference formula $C=2\pi r$ with the area formula $A=\pi r^2$, notice that both contain 2, π, and r, the 2 being an *exponent* in the area formula and a *coefficient* in the circumference formula.

42. Approximation in measurement. When a number is found by measuring, the last figure to the right is generally approximated, the others being exact. If the number 39.29 is obtained by measuring a segment, it does not mean that the real length is *exactly* 39.29, but that it is somewhere between 39.285 and 39.295.

Let 6.254 and 3.141 be two numbers, obtained by measurement in which the fourth figures were approximated. The work of finding the product of these numbers may then be arranged as follows:

$$\begin{array}{r}
6.25\dot{4} \\
\underline{3.14\dot{1}} \\
\dot{6}\ \dot{2}\dot{5}\dot{4} \\
250\ 1\dot{6} \\
625\ \dot{4} \\
\underline{18\ 76\dot{2}} \\
19.6\dot{4}\dot{3}\ \dot{8}\dot{1}\dot{4}
\end{array}$$

The result, 19.643814, is misleading. It expresses the product to eight figures, giving the impression of a high degree of accuracy. As a matter of fact some of these figures are meaningless, as will be seen from the following discussion:

Let the doubtful figures in the partial products be marked with dots placed over the figures. Since the last figure 4 in 6.254 is uncertain, the product of this 4 by any number is also uncertain. This makes the last figure on the right in each of the partial products uncertain. Hence, in the *sum* of the partial

products the last four figures on the right are meaningless and should be dropped. This leaves 19.64 as the *approximate* product, with the 4 in doubt.

Evidently, the product of the two given numbers is approximately 19.64, the four figures to the right having been left off because they have no meaning.

In multiplying numbers obtained by measurement the degree of precision in the product must be carefully considered. Otherwise this might lead to a misunderstanding. The degree of precision depends on the type of problem and the kind of instruments used in measuring. Making a circular flower bed, building a circular roundhouse for a railroad, and making a piston for an automobile engine call for various degrees of precision. The piston must be exact to the hundredth part of an inch, and the flower bed may be entirely satisfactory with a diameter one or two inches too long or too short. Failure to be exact in the making of the piston is fatal, but to apply the same precision in making a flower bed is not only misleading but useless.

43. Abbreviated multiplication saves time and effort. There is a short process of multiplying which avoids useless effort and saves time spent on multiplying and adding figures which are meaningless and of no value as far as the final product is concerned. The process is easily learned and it will be to your advantage to master it.

We have seen that in the product 19.643814, the figures 3814 have no meaning, because the figure 4 preceding 3814 is doubtful. To avoid writing meaningless figures proceed as in Examples 1 and 2, page 56.

1. Write down the two numbers to be multiplied	$\overset{\checkmark}{6}.\overset{\checkmark}{2}\overset{\checkmark}{5}\overset{\checkmark}{4}$ 3.141
Put a check mark over the 6 and multiply it by 1, the first number to the right in the multiplier	6
Put a check mark over the 2 and multiply both numbers checked by 4, but carry the 2 resulting from 4×5.	250
Put a check mark over the 5 and multiply the numbers checked by 1.	625
Put a check mark over the 4 and multiply the numbers checked by 3.	18762
Add, and place the decimal point by estimating	19.643

In the product the last figure to the right is in doubt, because the last figure to the right in every partial product is doubtful.

2. Find the product $3.52 \times 1.64 \times 7.15$.

Old Method:

$$
\begin{array}{r}
3.52 \\
1.64 \\
\hline
1\dot{4}\dot{0}\dot{8} \\
211\dot{2} \\
35\dot{2} \\
\hline
57\dot{7}\dot{2}\dot{8} \\
715 \\
\hline
2\dot{8}\dot{8}\dot{6}\dot{4}\dot{0} \\
57\dot{7}\dot{2}\dot{8} \\
404\dot{0}\dot{9}\dot{6} \\
\hline
41.2\dot{7}\dot{5}\dot{5}\dot{2}\dot{0}
\end{array}
$$

Abbreviated Method:

$$
\begin{array}{r}
\overset{\checkmark}{3}.\overset{\checkmark}{5}\overset{\checkmark}{2} \\
1.64 \\
\hline
14 \\
211 \\
352 \\
\hline
5.77 \\
7.15 \\
\hline
28 \\
58 \\
4039 \\
\hline
41.25
\end{array}
$$

Exercises

Find to the nearest third figure the following products, arranging your work as in Exercises 1 and 2.

1. 13.5×2.17.

Solution: Check the 1 and multiply it by 7, carrying the 2 from 7×3.

Check the 3 and multiply 13 by 1.

Check the 5 and multiply 135 by 2.

Then add the partial products. Place the decimal point by inspection.

$$\begin{array}{r} \checkmark\checkmark\checkmark \\ 13.5 \\ \underline{2.17} \\ 9 \\ 13 \\ \underline{270} \\ 29.2 \end{array}$$

2. 1.34×.68.

Solution:
$$\begin{array}{r} \checkmark\checkmark \\ 1.34 \\ \underline{68} \\ 10 \\ \underline{80} \\ .90. \end{array}$$

3. 16.2×4.18.

4. 20.1×6.07.

5. 9.60×16.5.

6. 180×3.14.

7. 12.4×3.14×14.6.

8. 240×2.76×17.1.

9. 6.37×19.1×46.2.

10. 1.36×0.47×3.12.

11. 1.39×2.42×1.47.

12. 4.12×0.41×2.51.

44. Abbreviated division. As in multiplication of numbers there is a waste of time in carrying the meaningless figures in dividing approximate numbers. The following example illustrates the process of abbreviated division in which the meaningless figures are omitted.

To divide 24.68 by 21.62 proceed as follows:

a. To make the divisor a whole number, multiply divisor and dividend by 100.

b. To place the decimal point, see how many times 2162 is contained in 2468. Write 1 in the quotient and place the decimal point after 1. Multiply the divisor by 1 and subtract.

c. Put a check mark over the 2, the first number at the right, in the divisor. The new trial divisor, 216, goes into the first remainder 306 once, leaving the remainder 90. Put 1 into the quotient.

```
   ✓✓✓  1.141
21.62)24.68
      21 62
       3 06
       2 16
         90
         86
          3
          2
          1
```

d. Check off the 6 in the divisor and try the new trial divisor 21 in the last remainder 90. This gives 4. Multiplying 21 by 4 and carrying 2 from the 4×6, we have 86. Subtract 86 from 90.

e. Check off the 1 and try 2 in the remainder 3, which gives 1. The quotient is 1.141, with the 1 at the right uncertain.

Exercises

Find the following quotients to as many figures as are contained in the dividend, using abbreviated division:

1. $\dfrac{81.52}{63.15}$.

Solution:

```
   ✓✓✓  1.291
63.15)81.52
      63 15
      18 37
      12 63
       57 4
       56 8
          6
          6
```

2. $\dfrac{42.67}{51.24}$.

Solution:

```
   ✓✓✓  0.833
51.24)42.67
      40 99
       1 68
       1 54
         14
         15
```

3. $\frac{72.8}{62.4}$. 5. $\frac{7.692}{8.526}$. 7. $\frac{12742}{47.58}$.

4. $\frac{489.90}{2.1360}$. 6. $\frac{979.7}{42.61}$. 8. $\frac{81.24}{9.963}$.

45. Problems involving the area of a circle. The determination of the value of π is one of the famous problems of geometry. It is known that Ahmes (1700 B.C.) took π equal to $(\frac{16}{9})^2$. Archimedes (287–212 B.C.) determined it to lie between $3\frac{10}{71}$ and $3\frac{10}{70}$. Ptolemy,

ARCHIMEDES

Archimedes was born in Syracuse, Sicily. He studied at the University of Alexandria and returned to Italy, where he lived the remainder of his life. Being a man of extraordinary ability, he has a record of wonderful achievements. You will find a number of interesting stories about his life, his works, and his death in books on the history of mathematics, *e. g.*, in Ball's *History of Mathematics*, pages 64–66.

who made astronomical observations at Alexandria from the years 125 to 150 A.D., calculated $\pi = 3.14166$. Vieta (1540–1603 A.D.) found the value to ten decimal places. Since then the computation has been carried to more than 700 decimal places. For our purposes 3.14 or 3.142 is usually sufficiently accurate. Taken to six figures π is equal to 3.14159.

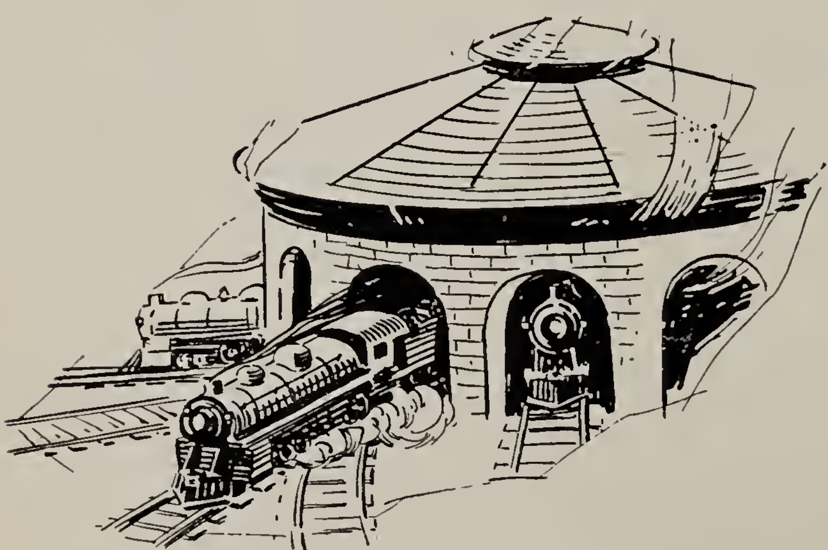
Exercises

In the following problems π is usually to be taken to three figures, as 3.14. If r contains more than three figures, π is to be taken to as many figures as are reliable figures in r^2. The reliability of the final results may be determined either by marking the doubtful figures in the multiplications and divisions, or by using the abbreviated processes explained in §§ 43, 44.

1. How much tin will be needed, not accounting for waste, to make the bottom of a pail 12 inches in diameter?

Solution:
$$\begin{aligned} A &= \pi r^2 \\ &= \pi \cdot 6^2 \\ &= 3.14 \times 36 \\ &= 113 \text{ approximately} \end{aligned}$$
$\therefore$ Area $= 113$ square inches.

Computation:
$$\begin{array}{r} 3.14 \\ 36 \\ \hline 19 \\ 94 \\ \hline 113 \end{array}$$



2. Find to three figures the area of a circle whose diameter is 7 feet.

3. How much ground is needed for building a roundhouse for a railroad if the radius is to be 52 feet?

4. Find the area of the circular surface of the piston of an automobile engine if the diameter is 3.75 inches.

5. What will it cost to resilver a circular mirror $2\frac{1}{2}$ feet in diameter at the rate of 50 cents a square foot?

6. What is the pressure of a piston head $1\frac{1}{2}$ feet in diameter, if the pressure is 236 pounds per square inch?

7. Find the area of a circular ring formed by two *concentric* circles (Fig. 45) whose inside and outside diameters are 8 inches and 10 inches respectively.

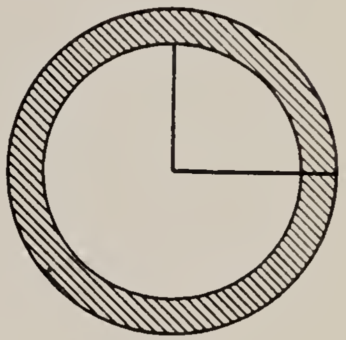


FIG. 45

8. A 3-foot walk surrounds a circular grass plot 40 feet in diameter. Make a sketch and find the number of square feet covered by the walk.

9. The cross section of a water main (Fig. 46) is a circular ring. Draw a sketch of the cross section and find the area if the pipe is $1\frac{1}{2}$ inches thick, and $1\frac{1}{2}$ feet in outside diameter.

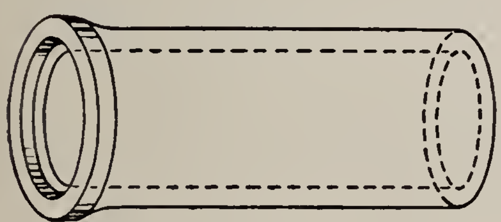


FIG. 46

10. A cow is tied with a rope to a stake which is driven at the mid-point of the longer side of a barn. If the barn is 14 feet by 18 feet and if the rope is 24 feet long, what is the area over which the cow can graze? Make a careful drawing before finding the area.

11. Find the areas of the circles having the following radii: 8 inches; 20 inches; 10.5 inches; 7.25 inches; 4.75 inches; $3\frac{2}{3}$ inches. Check the results of the last four circles by using the radii: $10\frac{1}{2}$, $7\frac{1}{4}$, $4\frac{3}{4}$, 3.67.

12. On centimeter squared paper draw a circle whose radius is 2 centimeters. Draw a square on the radius. Count the number of small squares within the circle, estimating the fractional parts. Count the small squares in the square on the radius. Divide the first number by the second. Tell why the quotient should be 3.14.

13. Find the radius of a circle whose area is 84 square feet.

Solution: $A=\pi r^2$

$$84=3.14r^2$$

$$\frac{84}{3.14}=r^2$$

or $\quad r^2=26.7$ approximately.

Extracting the square root of both members,

$r=5.1$ approximately.

Computation:

$$\begin{array}{r} 26.7 \\ 314\overline{)8400} \\ \underline{628} \\ 212 \\ \underline{188} \\ 24 \end{array}$$

$$\begin{array}{rr} & 5.1 \\ & \sqrt{26.7} \\ & \underline{25} \\ 101 & 170 \\ & 101 \end{array}$$

Find the radius of a circle whose area is

14. 68. **16.** 23.3

15. 251. **17.** 467.

18. Find to four figures the radius and area of the circle whose circumference is 39.29 feet.

Solution: Let r be the number of feet in the radius.

Take $\pi=3.141$, since the circumference is given to four figures.

Then $c=2\pi r$

Hence $39.29=2(3.141)r$

$$r=\frac{39.29}{2\times 3.141}=\frac{39290}{6282}=6.254$$

$$A=\pi r^2 \qquad =(3.141)\ (6.254)\ (6.254)$$

$$A=(19.64)\ (6.254)=122.9$$

$\therefore$ Area $=123$ square feet.

19. The circumference of the head of a drum is 7.6 feet. Find the number of square feet in the skin, making no allowance for overlapping.

Suggestion: Use $\pi=3.14$.

20. The circumference of a circle is 46.7 inches. Find the diameter to three figures.

21. The side of a square (Fig. 47) is 10.5 inches. Find the diameter and area of the circumscribed circle to three figures.

Suggestion: From the center of the circle draw radii to the end-points of one side. Use the theorem of Pythagoras to find the radius.

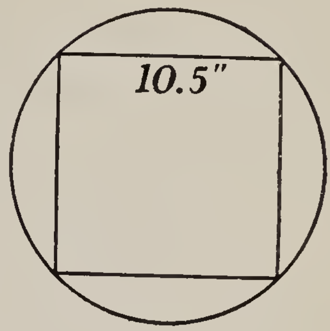


Fig. 47

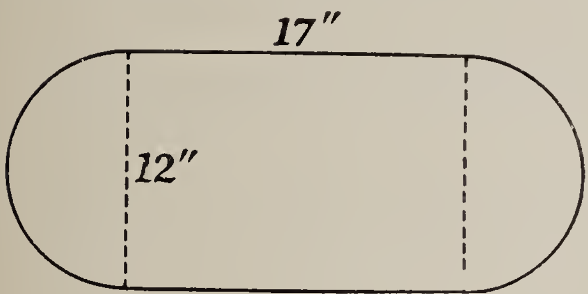


Fig. 48

22. How much tin is needed to make the bottom of a wash boiler (Fig. 48) whose ends are semicircles? Find the result to three figures.

23. Find to two figures the area of a circular ring (Fig. 49) if the radius of the inner circle is 6 inches and that of the outer circle is 11 inches.

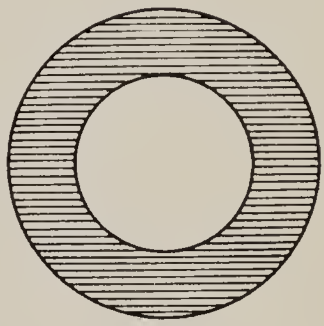


Fig. 49

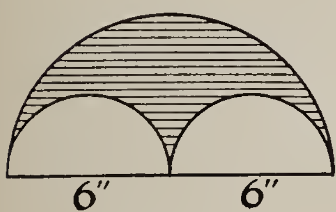


Fig. 50

24. Find to two figures the area of the surface included between the circles (Fig. 50), the radius of the large circle being 6 inches.

25. Find to two figures the area of the shaded surface (Fig. 51).

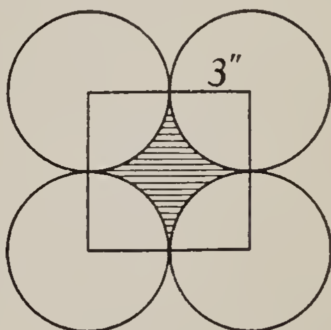


Fig. 51

26. A circle (Fig. 52) is inscribed within a square. Find the number of square feet in the shaded portion if the diameter of the circle is 8 feet.

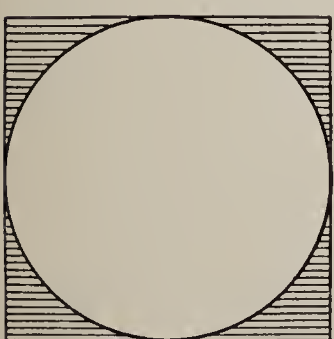


Fig. 52

27. The diameter of a piston of an automobile engine is $3\frac{1}{2}$ inches. Find the area of the circular surface of the piston.

46. What every pupil should know and be able to do. The following is a list of theorems and formulas studied in Chapter II:

1. *Opposite sides of a parallelogram are equal.*

2. *Opposite angles of a parallelogram are equal.*

3. *A diagonal divides a parallelogram into congruent triangles.*

4. *The area of a parallelogram is given by the formula:* $A=bh$.

5. *The area of a trapezoid is given by the formula:*
$A=\frac{h}{2}(a+b)$.

6. *The area of a triangle is given by the formula:*
$A=\frac{bh}{2}$.

7. *The area of a circle is given by the formula:*
$A=\pi r^2$.

Every pupil should be able to construct a parallelogram using only straightedge and compass; to make graphs of linear equations containing two literal numbers; to solve problems in areas by means of formulas and equations; and to use the abbreviated processes of multiplication and division.

47. Typical problems and exercises. Pupils should be able to solve the problems and exercises of the type given below:

1. Make a graph of each of the equations $A=6h$; $i=\frac{4}{100}p$.

2. The base of a parallelogram is $4x+7$, the altitude 5, and the area 75. Find x and the base.

3. Find the area of a trapezoid whose bases are 9 feet and 4 feet, and whose altitude is 6 feet.

4. Find the area of a circle whose radius is $3\frac{2}{3}$ inches.

5. Find the radius of a circle whose area is 76 square feet.

6. Find the area of the surface between two concentric circles whose radii are 12 inches and 9 inches respectively.

Solve the following equations and check each:

7. $\frac{7(x+6)}{2}=63.$ 8. $3.14r^2=82.4.$

9. Find the approximate quotient $\frac{39.29}{3.141}$, the last figures to the right being doubtful.

10. Find the approximate product

$$39.29\times3.141,$$

the last figures to the right being doubtful.

11. Write a paper on one of the following topics:

a. The use of the parallelogram in designs.

b. The degree of accuracy in multiplying numbers obtained by measuring.

c. The value of the formula in geometry.

d. The life and work of Archimedes or of some other famous mathematician mentioned in Chapter II.

CHAPTER III

AREAS OF SURFACES. VOLUMES OF SOLIDS

Rectangular Solids

48. What we are going to study in this chapter. The triangles, quadrilaterals, and circles which we have studied earlier in the course are called *plane* figures because they are figures all of whose points and lines lie in one and the same plane. The drawings illustrating important relations and properties of these figures have been made on the page of the textbook, on notebook paper, or on the blackboard. When the points and lines of a figure do not all lie in the same plane, relations, when represented by means of a drawing, are sometimes difficult to see for the beginner. Hence, in studying figures that are not plane, we shall first make models which actually represent the figures. They will help us to form mental images and to make drawings which we shall use to find the required relations.

In observing our surroundings we notice many objects of rectangular shape. Bricks, tanks, wagon boxes, freight cars, rooms, houses, and stores are rectangular. *Rectangular solids* are the first to be studied in this chapter. We shall make models of these solids, study the surfaces, and make use of some of the

known area formulas to measure the surfaces. Then we shall find a way of measuring the *content of the interior* enclosed by the surface. This gives the *volume* of the solid. Formulas will be worked out for finding areas of surfaces and volumes of solids. In doing this we shall use and review what we know about algebra and extend our knowledge of this subject.

In the same manner we shall study solids that are not rectangular, as the prism, cylinder, pyramid, cone, and sphere (Fig. 53).

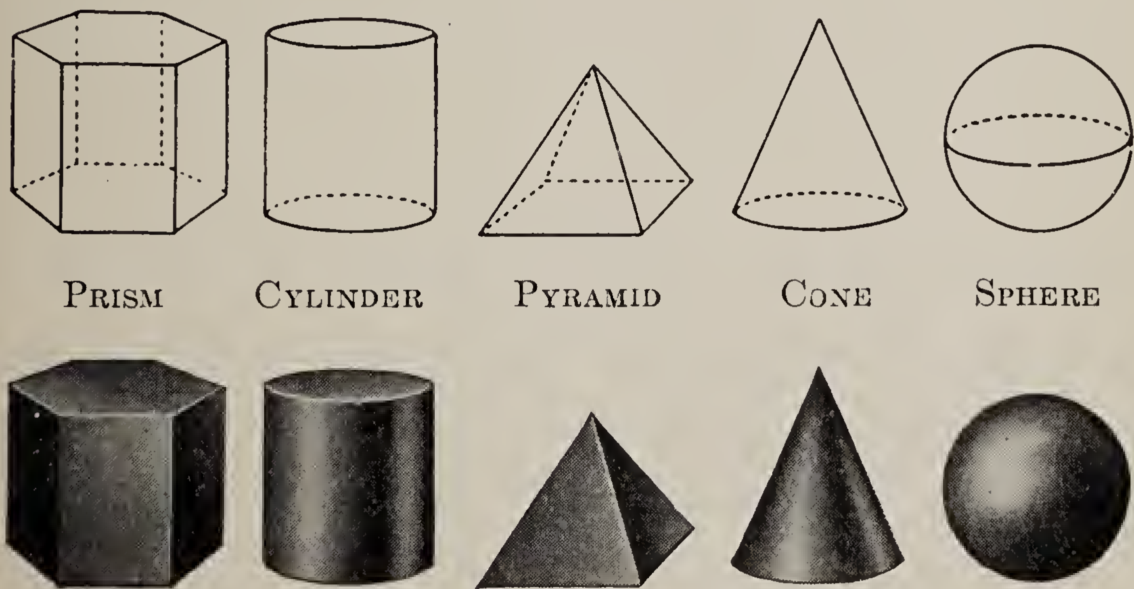


FIG. 53

The Cube

49. Making a model of a cube. On squared paper construct a diagram like the scale drawing shown in Fig. 54, using the dimensions indicated on the drawing. Notice that some of the lines are dotted, others are solid.

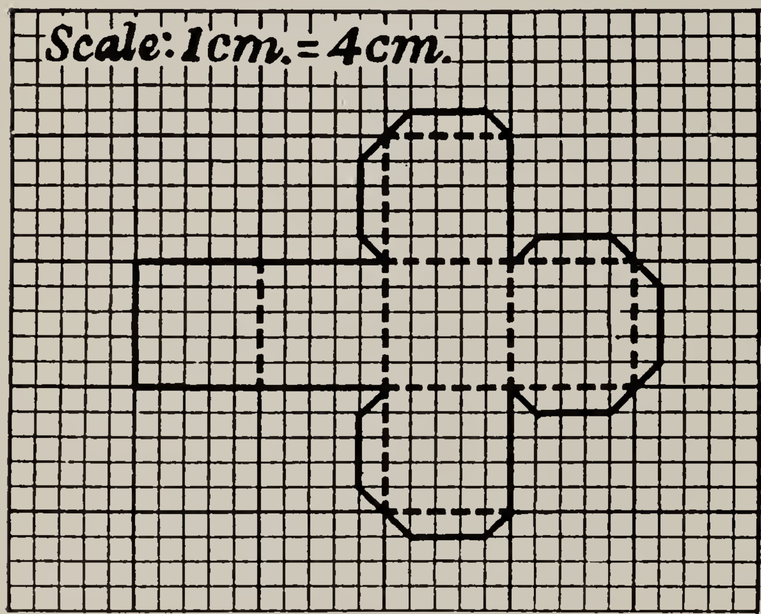


Fig. 54

Place your drawing on a sheet of heavy paper, or on cardboard.

Through all the corners of the drawing prick holes with a pin, and then make a copy on the cardboard.

Cut the cardboard along the solid lines, and crease it along the dotted lines. Paste the flaps, and close up the sides until a closed solid is formed. This is called a *cube*.

Exercises

1. How many faces (sides) has the cube?

2. Find the area of each face of your cube. Find the total area.

3. The lines of intersection of the faces are called *edges*, the points of intersection of the edges are the *vertices* of the cube. How many edges has a cube? How many vertices?

4. If the edge of a cube is 3 centimeters, find the area of each face; the total area.

5. Find the area of one face and the total area of a cube whose edge is 5 inches; 6 centimeters; e inches.

6. State a formula for finding the total area, t, of a cube in terms of the edge, e.

7. Using the formula of Exercise 6, find the total area of a cube whose edge is 5 inches.

Solution: $t = 6e^2$

$$\therefore t = 6 \times 25 = 150.$$

Similarly, find the total area of a cube whose edge is $3\frac{1}{2}$ centimeters; 16.2 inches; 36.14 centimeters; 4 feet; 8 inches.

8. On the cube point out: parallel lines, perpendicular lines, diagonals, right angles, squares, plane surfaces.

50. How to draw a cube. To make a diagram of a cube draw first the *base*, $ABCD$ (Fig. 55). Then draw lines AF, BE, CG, and DH in vertical position and equal in length. Finally draw $FEGH$.

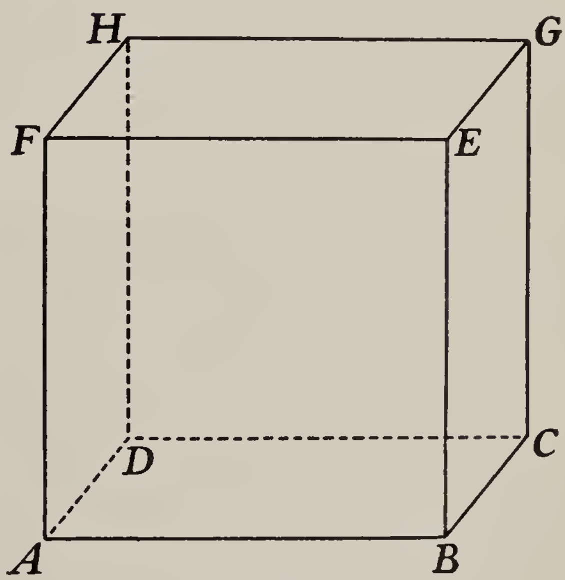


Fig. 55

The quadrilateral $ABCD$ may not at first look to you like a square but with a little training you will soon be able to see it as a square. If you have difficulty with seeing $ABCD$ as a square, this is because you are looking at it from the side, and not directly from the top down. With the square $ABEF$ you do not have this difficulty, because it is actually a square in the diagram, owing to the fact that you are looking at it directly and not from the side.

The *invisible* lines (the lines which cannot be seen when one is looking at the model) are drawn as *dotted* lines.

$ABCD$ is the *base* of the cube, because the cube stands on $ABCD$. However, any face may be considered as the base.

Exercises

1. Make several diagrams of cubes like the one shown in Fig. 55.

2. Note that line BE is perpendicular to AB and BC. It is said to be *perpendicular to the plane* of face $ABCD$.

Show similarly that the edges CG, DH, AF are perpendicular to the plane of $ABCD$.

In the classroom, locate lines which are perpendicular to planes.

3. The planes of opposite faces of a cube do not meet, however far they may be extended. They are said to be *parallel*. On the cube point out parallel planes. Point out parallel planes in the classroom.

4. Draw the diagonal AC in one of the faces of the cube (Fig. 56). Show that the formula for finding the length of AC in terms of the edge, e, is $d=\sqrt{2e^2}$ (§ 29).

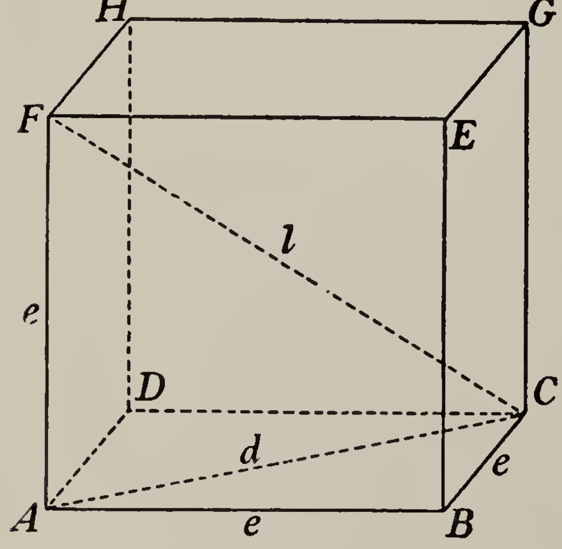


Fig. 56

5. Using the formula $d=\sqrt{2e^2}$ find, to four places of decimals, the length of the diagonal AC, if the edge of the cube is 4 centimeters; 7 centimeters; 3.68 centimeters.

Solution:

$$\begin{aligned} e&=4\\ e^2&=16\\ 2e^2&=32\\ d&=\sqrt{2e^2}=\sqrt{32}\\ d&=5.65. \end{aligned}$$

6. Draw the diagonal FC (Fig. 56) through the opposite vertices F and C. As in Exercise 4, find FC in terms of e and d; in terms of e only.

7. If $e=2.5$ find the length of the diagonal FC by means of the equation $l=\sqrt{3e^2}$ found in Exercise 5.

8. Find l if $e=3.2$; 6.5.

51. How to measure the space inclosed within the cube. Let the edge AB of a cube (Fig. 57) be 5 centimeters long.

Divide each edge of the cube into equal parts. In Fig. 57 the edge AB is divided into five equal parts. Imagine planes passed through the points of division, parallel to the base. Then the space enclosed in the cube is divided into five equal layers.

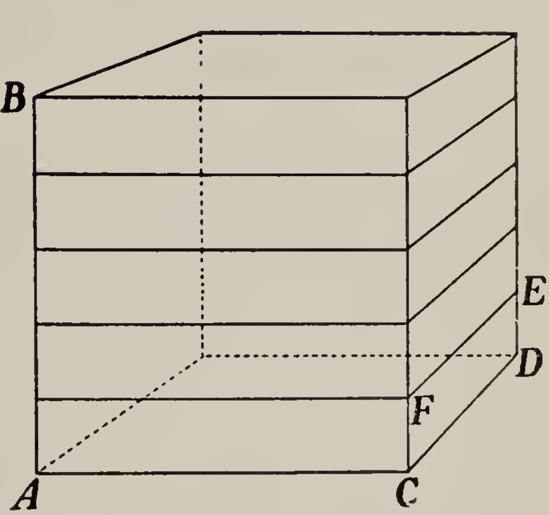


FIG. 57

In one of the layers (Fig. 58) imagine planes passed through the points of division of AC, parallel to plane $FCDE$. They divide the layer into five equal strips.

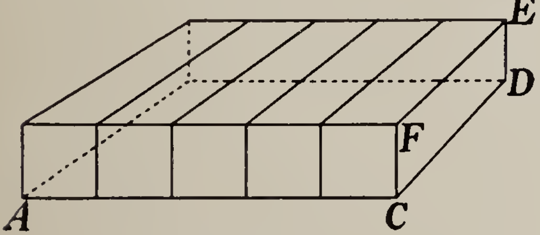


FIG. 58

Divide edge CD of the strip (Fig. 59) into five equal parts and through the points of division pass planes parallel to $CFGH$. They divide the strip into five equal cubes (Fig. 60).

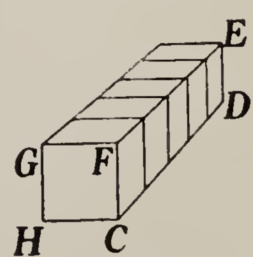


FIG. 59

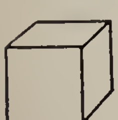


FIG. 60

Since each strip contains 5 centimeter cubes, each layer five strips,

and the cube five layers, it follows that the original cube contains $5\times5\times5$ *centimeter cubes.*

The number $5\times5\times5$ is a *measure* of the space enclosed within the cube. It is called the *volume* of the cube. In general, the volume of a given cube is the number of times that the given cube contains the *unit cube.*

Just as 5×5 is written briefly 5^2 (read "5-square"), the product $5\times5\times5$ is written in the brief form 5^3, (read "5-cube").

Hence, the volume of the cube (Fig. 57) is 5^3, or 125.

Similarly, the volume of a cube with the edge e, is $e\times e\times e$ or e^3, and may be found from the formula

$$v=e^3.$$

Exercises

1. Measure the edge of a given cube and find the volume.

2. Using the formula, find the volume of a cube whose edge is 3 centimeters; 4.5 centimeters; 6.2 centimeters; $3\frac{1}{2}$ centimeters; 3.14 centimeters.

3. State the meaning and value of each of the following cubes: 1^3, 2^3, 3^3, 4^3, ..., 10^3.

4. Make a table of cubes of the whole numbers from 1 to 10. Memorize the table.

5. Find the number of cubic inches contained in a cubic foot; the number of cubic feet contained in a cubic yard. Change to cubic inches: 5 cubic feet; 20 cubic feet; $12\frac{1}{2}$ cubic feet.

6. What part of a cubic foot is a cubic inch?

7. What part of a cubic yard is a cubic foot?

8. At 95 cents per 1000 cubic feet what is the cost of 2576 cubic feet of gas?

9. Find the value of x^2; x^3; $2x^3$; $3x^3$; $(3x)^3$; $3x^2$; $6x^3$; $8x^2$; when $x=2$; 2.5; $\frac{3}{2}$.

10. Find the value of x^3+3x^2+3x+9 for $x=\frac{3}{2}$.

Solution:

$$\begin{aligned} &x^3+3x^2+3x+9 \\ &=(\tfrac{3}{2})^3+3(\tfrac{3}{2})^2+3(\tfrac{3}{2})+9 \\ &=\tfrac{27}{8}+3(\tfrac{9}{4})+3(\tfrac{3}{2})+9 \\ &=\tfrac{27}{8}+\tfrac{27}{4}+\tfrac{9}{2}+9 \\ &=\tfrac{189}{8}=23\tfrac{5}{8}. \end{aligned}$$

11. Find the value of a^3+6a^2+4a-1 for $a=2$.

12. Find the exact value of $2m^3+5m^2-2m-3$ for $m=3.5$.

13. Find the exact value of $4t^3-t^2+6t-1$ for $t=2\frac{2}{3}$.

14. Find the exact value of $8b^3+5b^2+\frac{3}{4}-3b$ for $b=\frac{1}{6}$.

15. If the volume of a cube is 27 cubic inches, find the edge by means of an equation.

Solution:

$$\begin{aligned} v&=e^3 \\ v&=27 \\ \therefore e^3&=27 \\ \therefore e&=3. \end{aligned}$$

16. Find by means of an equation the edge of a cube if the volume is 8; 64; 216; 729.

52. The meaning of cube root. Just as we called the number 2 the *square root* of 4 because $2^2=2\times 2=4$, we shall call the number 2 the *cube root* of 8 because $2^3=2\times 2\times 2=8$. In general, the *cube root* of a number is one *of the three equal factors whose product is equal to that number.*

If a number is an *exact cube*, as 8, 27, 64, the *exact cube root* can be found by trial. If a number, as 7, is not an exact cube, the exact cube root is written $\sqrt[3]{7}$, and is read "cube root of 7."

53. Powers. The products $a \times a$, $a \times a \times a$, have been written in the brief forms a^2 and a^3 respectively. Similarly, we shall write $a \times a \times a \times a = a^4$, $a \times a \times a \times a \times a = a^5$, etc., read *a-fourth, a-fifth,* etc. The products a^2, a^3, a^4, etc., are *powers* of a.

Exercises

If $a=2$, $b=1$, $c=3$, find the value of each of the following. Arrange the work as in Exercise 10 (§51).

1. $a+b^2+c^3$.

2. $\dfrac{a^3-b^3+c^3}{2a^2}$.

3. $\dfrac{b^2+a^2-c.}{a+b+c}$

4. $ab+\dfrac{a^2}{c}+\dfrac{2}{c^2}$.

5. $2a^3+3a^2-5c$.

6. $\dfrac{a^3-2b+c}{3c-2a}$.

7. $\dfrac{a^2+2ab+b^2}{ab}$.

8. $\dfrac{a^4+a^2+3}{b^2+c^2}$.

Solve the equations in Exercises 9 to 14.

9. $2x^3=128$.

10. $4x^3=108$.

11. $3x^3-5=76$.

Solution:

$$\begin{aligned} 3x^3-5&=76 \\ 3x^3&=81 \\ x^3&=27 \\ x&=3. \end{aligned}$$

12. $2x^3+10=26$.

13. $5x^3-20=300$.

14. $3x^3-8=73$.

15. The number of gallons in a tank is approximately 7.5 times as great as the number of cubic feet. Find how many gallons there are in a tank whose volume is $4\frac{1}{2}$ cubic feet; $5\frac{1}{2}$ cubic feet; $8\frac{1}{2}$ cubic feet.

THE RECTANGULAR BLOCK

54. Rectangular solids. Cubes, boxes, cakes of ice, and many other objects with six faces, have each face of rectangular form. They are called *rectangular solids* or *blocks*.

To make a rectangular block from cardboard you may use as a pattern the diagram in Fig. 61. First draw this diagram on squared paper, and then follow carefully the directions for making the model for a cube that were given in §49.

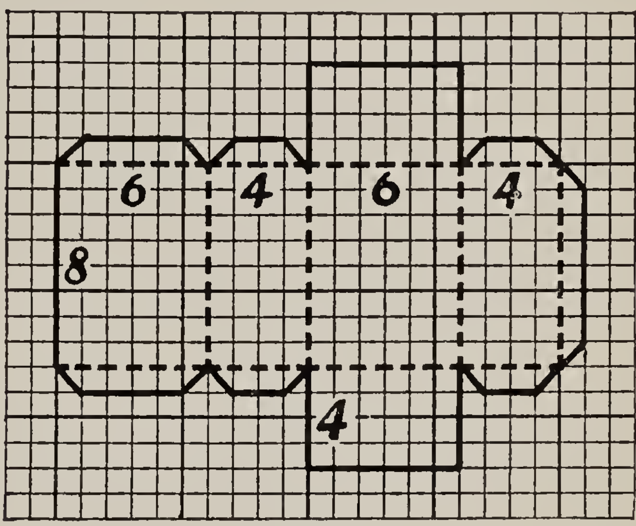


FIG. 61

EXERCISES

1. On your model of the rectangular block, point out parallel lines, parallel planes, lines perpendicular to planes, diagonals.

2. How many edges has the rectangular block? How many vertices?

3. Make several drawings of a rectangular block shown in Fig. 62.

55. A formula for finding the area of the surface of a rectangular block. Let a, b, c be the dimensions of a rectangular block (Fig. 62). Show that the area of the total surface may be found from the formula $t = 2ab + 2bc + 2ca$.

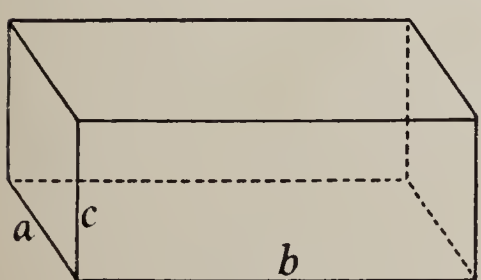


FIG. 62

Exercises

Using the formula $t=2ab+2bc+2ca$, find the area of rectangular blocks having the following dimensions:

1. $a=4$.	$b=1$.	$c=3$.
2. $a=1.3$.	$b=4.6$.	$c=2.5$.
3. $a=2\frac{1}{2}$.	$b=1\frac{2}{3}$.	$c=4\frac{1}{5}$.

4. How many square inches of copper lining are required to line the sides and bottom of a rectangular tank 18 inches long, 8 inches high, and 9 inches wide?

56. How to find the volume of a rectangular block. Let the dimensions of a rectangular block (Fig. 63) be

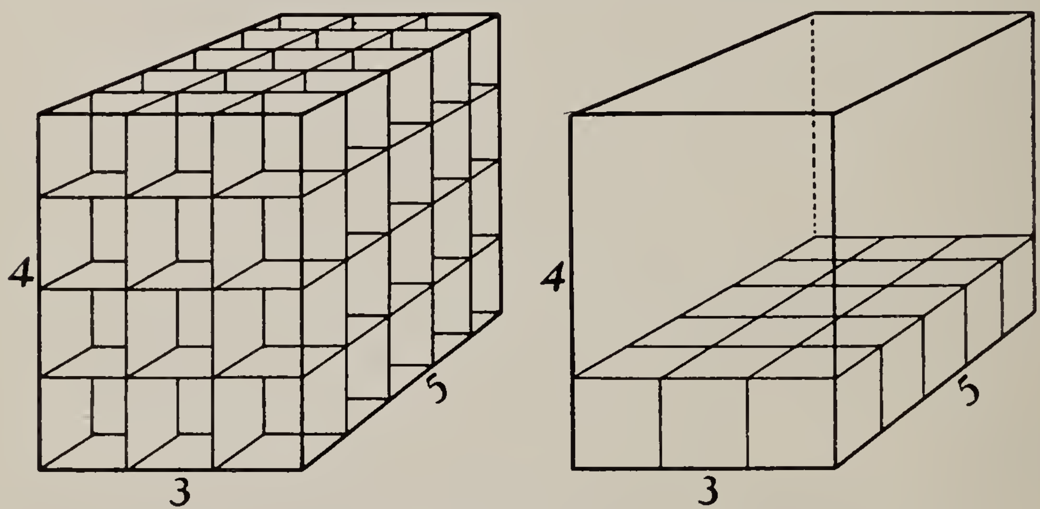


Fig. 63

3 centimeters, 4 centimeters, and 5 centimeters respectively.

We can divide the solid into four layers, each layer into three strips, and each strip into five cubes.

Hence the solid contains $4\times3\times5$ centimeter cubes, and the volume is said to be $4\times3\times5$, or 60.

This illustrates that the *volume of a rectangular block is the product of the three dimensions.*

If we denote the length by l, the width by w, the height by h, and the volume by v, we have the formula

$$v = l \times w \times h,$$

$$\text{or } \mathbf{v = lwh.}$$

Exercises

1. Measure the edges of a rectangular block and find the volume.

Suggestion: Use the formula $v = lwh$.

2. A classroom is 24 feet by 17 feet by 12 feet. Find how many cubic feet of air space it contains.

3. If a cubic foot of ice weighs 58 pounds, what is the weight of a piece of ice 10 inches by 16 inches by 18 inches?

Suggestion: $w = \dfrac{10 \times 16 \times 18 \times 58}{12 \times 12 \times 12}$.

Change the fraction to the simplest form.

4. How many cubic feet must be excavated to make a ditch $2\frac{1}{2}$ feet wide, 3 feet deep, and 120 feet long?

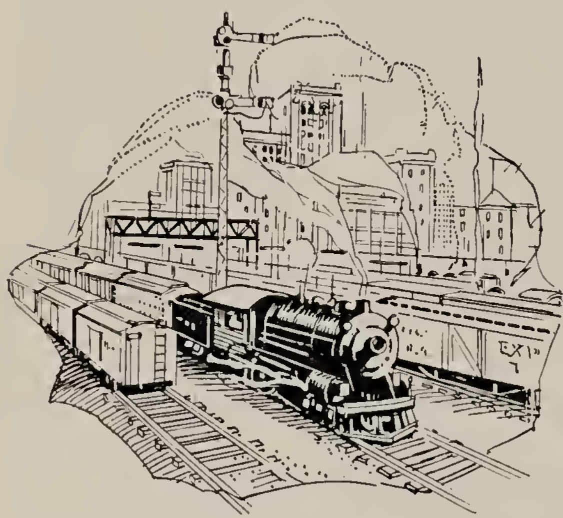
5. How many bars of soap can be packed into a box 2 feet by 2 feet by 1 foot, if a bar of soap when wrapped is $2\frac{1}{2}$ inches by $4\frac{1}{2}$ inches by $1\frac{1}{2}$ inches?

6. During a storm one-half inch of rain fell. How many gallons of water fell on a lot 25 feet by 175 feet, if a cubic foot holds approximately 7.5 gallons of water?

7. A coal bin 12 feet wide and 14 feet long is filled with coal to a depth of 5 feet. If a cubic foot of coal weighs 63 pounds, how many tons of coal does the bin contain?

8. Mr. Johnson's coal bin is 8 feet wide and 12 feet long. If he orders 15 tons of coal, to what height should the bin be filled?

9. A freight car 12 yards long and 3 yards wide is filled with wheat to a depth of 2 yards. If a cubic foot holds $\frac{4}{5}$ of a bushel how many bushels does the car contain?



10. We have seen that the volume of a rectangular solid (Fig. 64) may be found from the formula $v = abc$.

Since $abc = a(bc)$ and since bc is equal to the area of the base, it follows that *the volume of a rectangular solid is equal to the product of the number of units of length in the altitude by the number of surface units in the base.*

Find the volume of a rectangular solid whose base is 24 square inches and whose altitude is 6 inches.

11. If a load of gravel is 1 cubic yard, how many loads are required to make a road 3 miles long, if spread 9 feet wide and 8 inches deep?

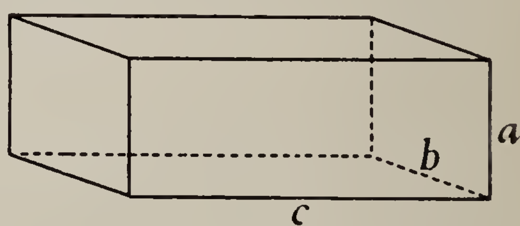


Fig. 64

12. An L-shaped corner building (Fig. 65) is to be erected having frontages of 60 feet and 48 feet on the intersecting streets. Find the cost of excavating a cellar 12 feet deep and 26 feet wide at 40 cents a load, or cubic yard.

13. A swimming tank 36 feet wide and 48 feet long is filled with water to a depth of 5 feet. If we allow 7.5 gallons of water per cubic foot, how many gallons of water are there in the tank?

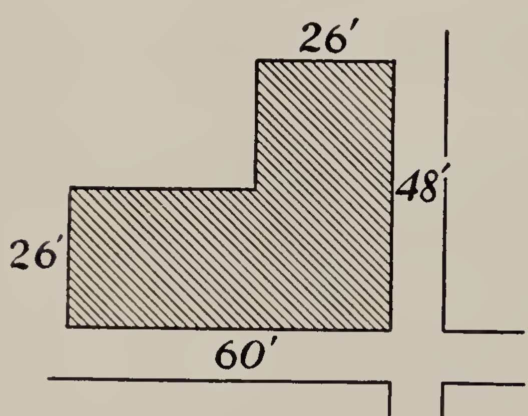


Fig. 65

14. Let $v = abc$. Find c if $v = 400$, $a = 4$, $b = 10$; if $v = 144$, $a = 8$, $b = 10$.

15. A rectangular tank 12 feet long and 10 feet wide contains 400 gallons of water. Find the depth.

Suggestion: Change the number of gallons to cubic feet.

16. A cubic inch of gold is to be rolled into gold leaf $\frac{1}{1000}$ of an inch thick. How many square inches of gold leaf will it make?

17. How many tons of hard coal are there in a bin 11 feet by $9\frac{3}{4}$ feet when the pile is $4\frac{1}{2}$ feet high, assuming that a ton of hard coal occupies a space of 35 cubic feet?

18. A schoolroom is 40 feet long, 35 feet wide, and 12 feet high. If we allow 450 cubic feet of air for each pupil, how many pupils may be accommodated in the room?

19. A *board foot* is a piece of wood 1 foot long, 1 foot wide, and 1 inch thick (Fig. 66). Lumber may be measured in board feet. Lumber *less* than one inch in thickness is figured as if it were an inch thick. Material *more* than an inch thick is measured according to actual thickness in inches and fractions of an inch. Find the number of board feet in a 10-foot board 2 inches by 6 inches.

Solution:

Denoting by t the number of inches in the thickness;
by w the number of feet in width;
by l the number of feet in length;
and by b the number of board feet;
we have

$t=2''$

$w=6''=\frac{6}{12}'$

$l=12'$

and $b=2\times\frac{6}{12}\times 10$

$\therefore b=10.$

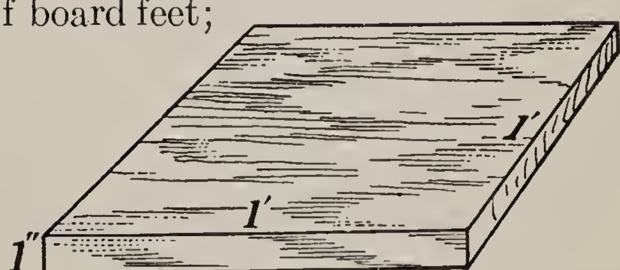


FIG. 66

20. Find the number of board feet in a beam 3 inches by 4 inches, 14 feet long.

21. If b is the number of board feet, t the thickness in inches, w the width in inches, and l the length in feet, show that

$$b=t\times\frac{w}{12}\times l$$

22. Using the formula in Exercise 21, find the number of board feet in a plank 2 inches thick, 12 inches wide, and 16 feet long.

23. Find the number of board feet in a piece of lumber $1\frac{3}{4}$ inches thick, 6 inches wide, and 12 feet long.

24. A boy wishes to make a bookcase with four shelves, each of which is to be 24 inches by 8 inches. The two sides are to be 4 feet by 9 inches. How much material will he need, and what will it cost at 12 cents a board foot?

Find the number of board feet in each of the following:

25. 6 planks 2 inches thick, 12 feet long, 8 inches wide.

26. 14 planks $1\frac{3}{4}$ inches thick, 10 feet long, 12 inches wide.

27. 9 beams 4 inches by 4 inches and 8 feet long.

28. Lumber is usually priced *per thousand* (M) board feet. What is the price of lumber per board foot if sold at \$45.00 per thousand (M) feet?

Find the cost of each of the following:

29. 2050 feet of pine flooring at \$65.00 per M.

30. 5000 feet of oak at \$85.00 per M.

Write each of the following in the simplest form:

31. $a \cdot a \cdot a$.

32. $a^2 \cdot a$.

33. $(3y)^3$.

34. $5a \cdot 2a^2$.

35. $3x \cdot 5x^2$.

36. $6m^2 \cdot 2m$.

37. $2a \cdot 3a \cdot 5a$.

38. $x \cdot 9x \cdot 2x$.

39. $(2a)^3 + (3b)^3$.

The Prism

57. How to draw and make a prism. If through the corners of a polygon P (Fig. 67) parallel lines, AB, CD, EF, etc., are drawn, not in the plane of the polygon, and if a plane Q is passed parallel to the plane of

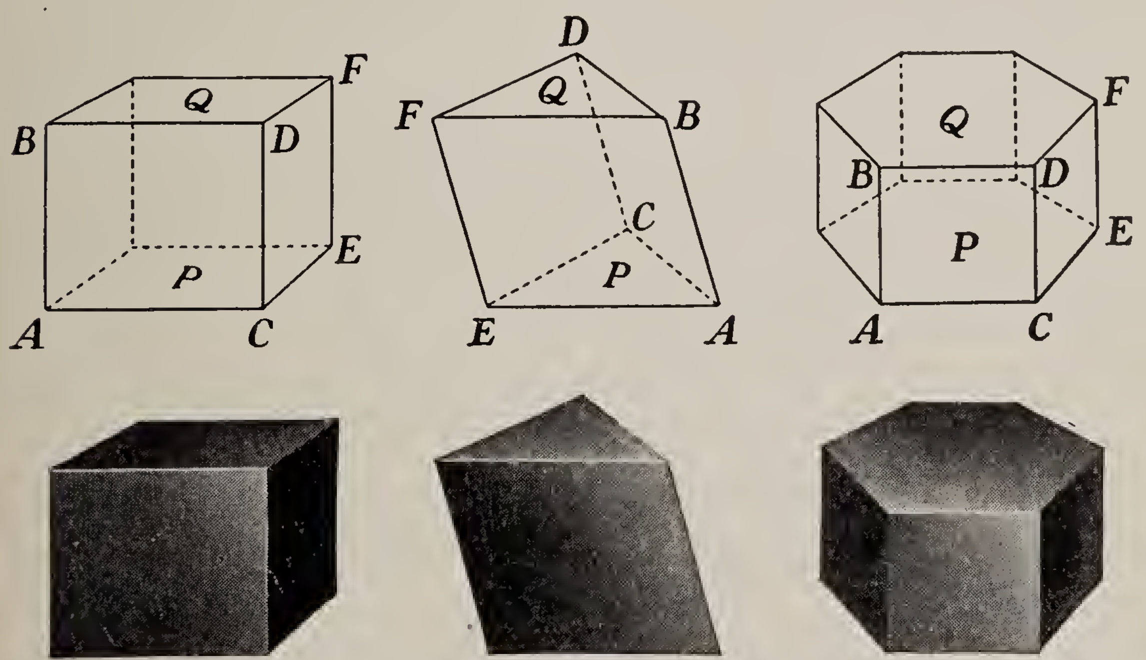


Fig. 67

the given polygon, a *prism* is formed. Polygons P and Q are the *bases*, and the other faces, as $ABDC$, are the *lateral faces* of the prism. The lateral faces are parallelograms. Note that the rectangular solids studied in §§48 to 56 are prisms. Tell why they are prisms.

The lines of intersection of the planes are the *edges* of the prism. The intersections AB, CD, EF, etc., of the lateral faces are called *lateral* edges. When the lateral edges are perpendicular to the planes of the bases the prisms are *right* prisms. Prisms are said to be triangular, quadrangular, or hexagonal, according as the bases are triangles, quadrilaterals, or hexagons.

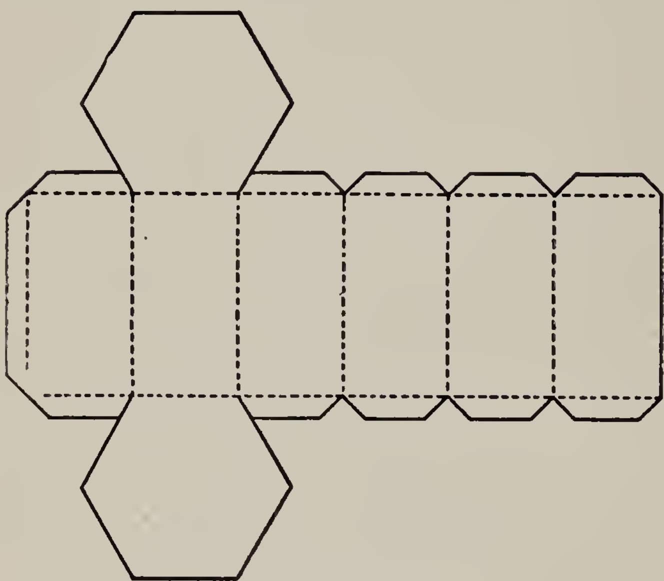


Fig. 68

Draw a pattern (Fig. 68) and use it to make a model of a hexgonal right prism.

Practice making figures like those in Fig. 67 until you can make a good drawing of a prism.

Exercises

1. The sides of the base of a right prism are a, b, c, and d. The altitude is e. Make a sketch of the prism. State a formula for finding the lateral area.

Using the lateral area formula $L=e(a+b+c+d)=ep$, where p is the perimeter of the base, find the lateral area of each of the right prisms below.

	a	b	c	d	e	L
2.	2	5	3	10	7	exact value
3.	$1\frac{1}{2}$	$2\frac{3}{4}$	$1\frac{2}{3}$	3	$8\frac{1}{2}$	exact value
4.	4.3	2.6	5.4	3.8	12	value to three figures
5.	4.23	2.12	4.01	3.64	9.32	value to three figures

58. A formula for finding the volume of a right prism. As in the rectangular solids, the volume of the prism, as ABC-F (Fig. 69), is found as follows:

Divide the altitude CD into h equal parts. Then divide the prism into h equal layers by drawing planes which are parallel to the base and pass through the points of division. On each layer we can place as many unit cubes as there are unit squares in the base. Thus, if the altitude of the prism is h units long and if the area of the base is b square units, the prism has a content equal to $b \times h$ cubic units, *i.e., the volume of a right prism is equal to the product of the base by the altitude.*

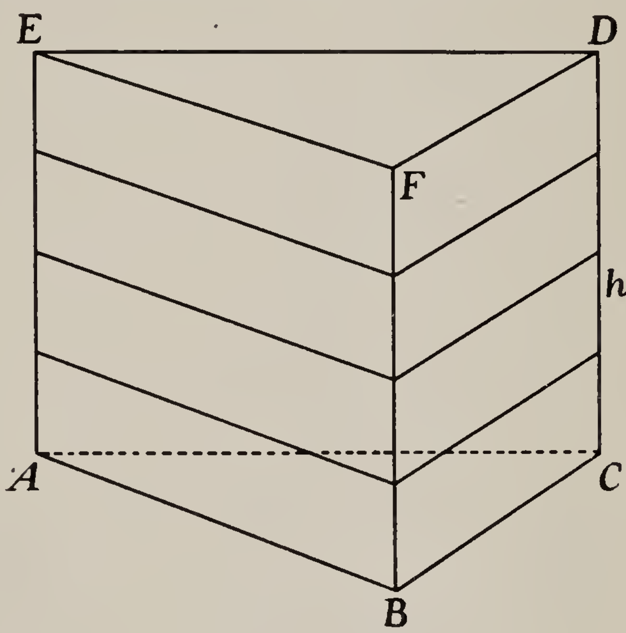


Fig. 69

Briefly we say that

$$\mathbf{v = bh.}$$

Exercises

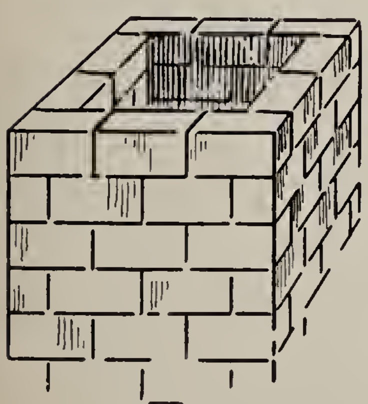


Fig. 70

1. By means of the formula $v = bh$ find the volume of a triangular right prism the area of whose base is 18 square inches and whose altitude is 12 inches.

2. Find the volume of a right prism whose altitude is 13.4 inches and the area of whose base is 65.3 square inches.

3. To order material for building a chimney (Fig. 70), John's father must know

the amount of masonry required. The chimney is to be 35 feet high, 24 inches wide, and 18 inches deep. It has a flue 12 inches by 14 inches. Find the number of cubic feet in the masonry.

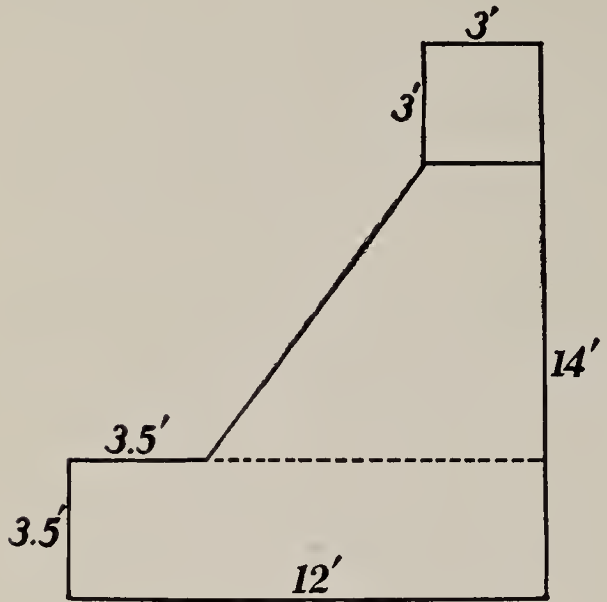


Fig. 71

4. The base of a triangular right prism is a right triangle whose sides are 3 inches, 4 inches, and 5 inches. The altitude is 10 inches. Find the lateral area; the volume.

5. The diagram (Fig. 71) represents the cross section of a concrete wall. If the wall is 120 feet long, how much concrete will be required to build it?

6. Suppose you find on a winter morning that the snow on your sidewalk is 4 inches deep. How many cubic feet of snow do you have to remove to clear your walk if it is 50 feet long and 4 feet wide?

The Cylinder

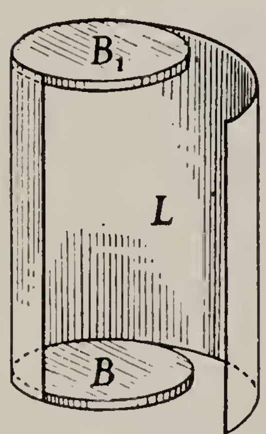


Fig. 72

59. How to make a model of a cylinder. Circular cans, water pipes, steam boilers, lawn rollers, tanks, and many other objects are of cylindrical shape. To make a model of a *right circular cylinder*, cut, from card board, two congruent circular disks (Fig. 72) and then paste a rectan-

gular strip L whose length is equal to the circumference of the circles so as to form the lateral surface.

Make several drawings of a cylinder (Fig. 73).

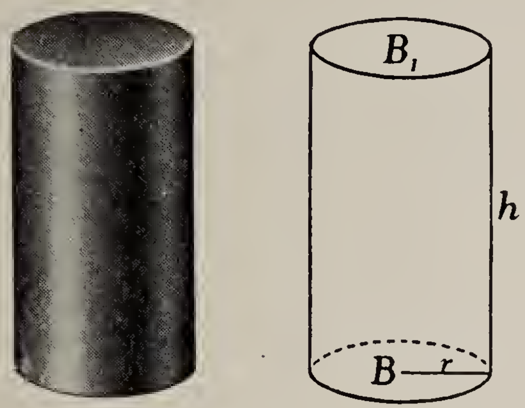
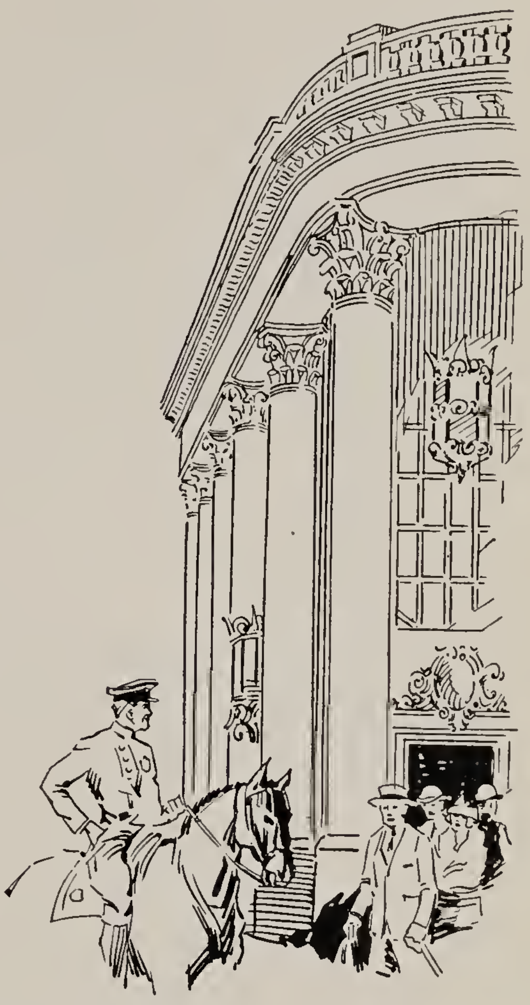


FIG. 73



The two congruent circles B and B_1 (Fig. 72) which enclose the flat surfaces of the cylinder are called *bases;* the perpendicular distance, h, between the bases is the *altitude;* and the curved surface, L, is the *lateral surface* of the right circular cylinder. The bases lie in parallel planes.

60. A formula for finding the lateral area of a right circular cylinder. The area of the lateral surface (Fig. 74) may be found by rolling the cylinder on a plane surface R.

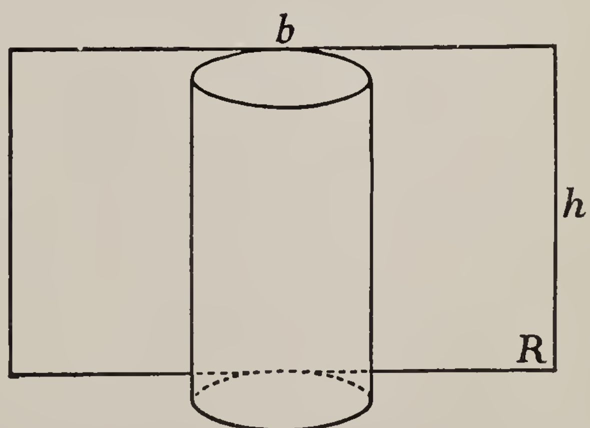


FIG. 74

The lateral surface will just cover a rectangle

whose base b is equal to the circumference of the base of the cylinder and whose altitude h is equal to the altitude of the cylinder.

This shows that the *lateral area of a right circular cylinder is equal to the circumference of the base multiplied by the altitude.*

Denoting the number of square units in the lateral surface by L, the number of units in the altitude by h, and the number of units in the radius of the base by r, we have

$$\mathbf{L = 2\pi rh.}$$

Exercises

1. How much does it cost to paint the lateral surface of a silo 14 feet in diameter and 25 feet high, at the rate of \$2.25 per 100 square feet?

Solution: $L = 2\pi rh$

$d = 14$

$r = 7$

$h = 25$

Since the cost is figured in dollars and cents, use $\pi = 3.14$.

$$L = 2(3.14)\times 7\times 25$$

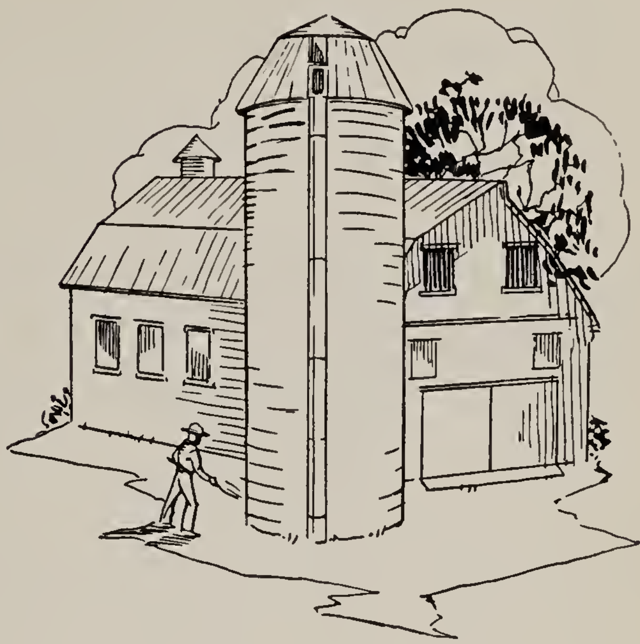
$$\therefore \text{Cost} = \frac{2\overset{1.57}{\cancel{(3.14)}}\times 7\times \cancel{25}\times 2.25}{\underset{\underset{2}{\cancel{4}}}{\cancel{100}}}$$

or Cost $= \$24.71$.

Computation:

$$\begin{array}{r} 1.57 \\ 7 \\ \hline 10.99 \\ 2.25 \\ \hline 54 \\ 2\ 19 \\ 21\ 98 \\ \hline 24.71 \end{array}$$

2. Find the number of square inches of sheet iron in a stove pipe 6 inches in diameter and 6 feet long, not allowing for locking.



3. Find to three figures the lateral area of a cylinder whose altitude is 9 inches and the radius of whose base is 5 inches.

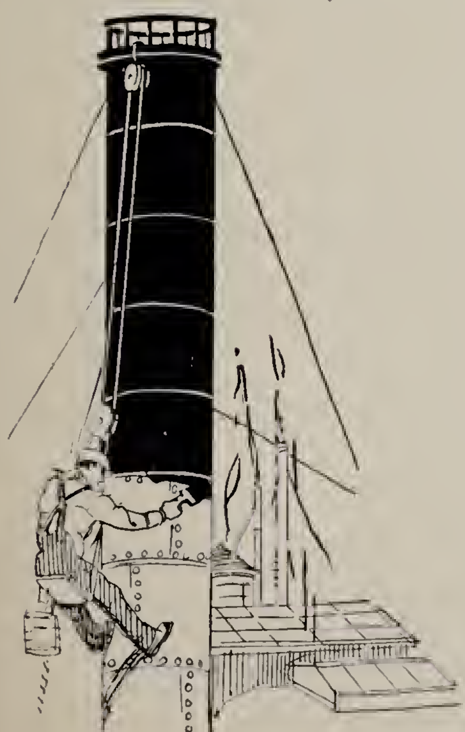
4. Find the radiating surface of a hot-water pipe 3 inches in diameter and 24 feet long.

5. How many square feet of ma erial are used in making a cylindrical tank 12 inches in diameter and $3\frac{1}{2}$ feet high?

6. Find the lateral area and total area of each of the following cylinders:

Radius	18 ft.	12.5 in.	3 cm.	3 in.
Height	9 ft.	5 ft.	20 cm.	15 ft.

7. How much does it cost to paint a cylindrical chimney 25 feet high and 2.5 feet in diameter at a rate of $2.75 per 100 square feet?



8. Find the cost of painting a cylindrical gasoline storage tank 22 feet high and 25 feet in diameter at a rate of $2.25 per 100 square feet.

9. When making application for an automobile license a man is told to compute the horse power of the engine by means of the formula

$$\text{h.p.} = \frac{d^2 n}{2.5}$$

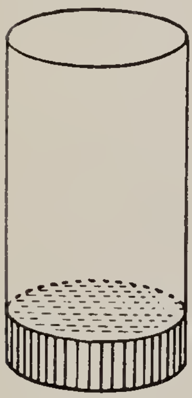
where d is the length of the diameter of the piston and n the number of

cylinders. Find to three figures the horse power of a four-cylinder engine the diameter of whose piston is $3\frac{1}{2}$ inches.

10. What is the horse power of an eight-cylinder engine the diameter of whose pistons is $4\frac{1}{2}$ inches?

61. A formula for finding the volume of a right circular cylinder. We may derive a formula for finding the volume of a cylinder as follows:

Since the base (Fig. 75) contains πr^2 unit squares, we can place πr^2 unit cubes on it, forming a layer one unit high. If the altitude is h units long, we shall have h such layers in the cylinder. Hence the cylinder will contain $(\pi r^2)h$ unit cubes.



h

Fig. 75

Therefore the volume of a right circular cylinder may be found from the formula

$$\mathbf{v = \pi r^2 h},$$

where r is the radius of the base, h the altitude, and $\pi = 3.142$. The number of figures to be taken in π in a given problem depends on the degree of accuracy to be attained.

Exercises

1. A gallon contains approximately 231 cubic inches of water. How many gallons of water does a steam boiler (Fig. 76) hold that is 12 feet long and 3 feet in diameter?

Solution:

$$v = \pi r^2 h$$
$$\pi = 3.14$$
$$r = \frac{3}{2}$$
$$h = 12$$
$$\therefore v = \frac{(3.14)\times 3\times 3\times \overset{3}{\cancel{12}}}{\cancel{2}\times\cancel{2}}$$
$$\therefore v = 84.7.$$

Computation:

3.14
27
2 19
6 28
84.7

Hence the boiler contains approximately 84.7 cubic feet of water.

To find the number of gallons, n, contained in the tank, we find the number of gallons in one cubic foot and multiply the result by the number of cubic feet.

$$\therefore n=\frac{\overset{\overset{1.1}{\cancel{12.1}}}{\cancel{(84.7)}}\times 12\times 12\times \overset{4}{\cancel{12}}}{\underset{\underset{\cancel{11}}{\cancel{77}}}{\cancel{231}}}$$

$$=(1.1)\ (12)\ (12)\ 4$$

$$\text{or } n=633.6$$

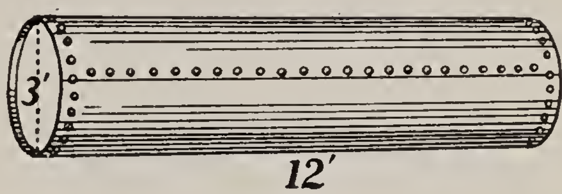


FIG. 76

2. Find the volume of a cylindrical gas tank 50 feet in diameter and 36 feet high.

Suggestion: Take $\pi=3.142$.

3. A farmer wishes to make a concrete roller 2 feet in diameter and 5 feet wide. How many cubic feet of concrete will he need?

4. Find the volume of each of the following cylinders:

Altitude	12 in.	8 yd.	$16\frac{1}{2}$ ft.	25 in.
Radius	6 in.	4 yd.	2 ft.	$6\frac{1}{2}$ in.

5. How many gallons will a hot-water tank hold which is 1 foot in diameter and 5 feet long?

6. Find the capacity of a stand pipe 12 feet in diameter and 45 feet high.

7. The inside diameter of a silo is 12 feet, and the height 30 feet. How much silage does it hold?

8. Find the depth of a cylindrical cistern 6 feet in diameter holding 600 gallons.

Suggestion: Change 600 gallons to cubic feet.

9. An iron cylindrical pillar with a diameter of 6 inches is 12 feet long. Find the weight, if a cubic foot weighs 440 pounds.

10. A cylindrical bar of cast iron is 3 inches in diameter and 10 feet long. Find the volume and weight if the weight of cast iron is 450 pounds for a cubic foot.

11. Find the volume of iron in a pipe whose outside and inside diameters are respectively 6 inches and 5 inches, and whose length is 8 feet.

12. The external diameter of a water main is 4 feet and the pipe is 1 inch thick. If a cubic inch of pipe weighs .26 pounds, what is the weight of 10 feet of pipe?

13. Find the number of gallons of oil contained in a cylindrical tank car 30 feet long and 6 feet in diameter.

14. The inside diameter of a cylindrical silo is 16 feet and the height is 25 feet. How many tons of silage will it hold if a cubic foot of silage weighs 45 pounds?

15. A cubic foot of copper is to be drawn into a wire $\frac{1}{10}$ of an inch in diameter. How long is the wire when drawn?

Find the value of:

16. $\pi r^2h+rh+h$, when $r=2$, $h=2.5$.

Solution:
$$\begin{aligned}\pi r^2h+rh+h &= (3.14)4(2.5)+2(2.5)+2.5\\ &=31.4+5+2.5\\ &=38.9.\end{aligned}$$

17. $2x^3+5x^2+3x$ when $x=1.2$.

18. $3m^3+m^2+7m$ when $m=.5$.

Multiply as indicated and check by substituting values for the letters:

19. $5(x+2)$.

Solution:
$5(x+2)=5x+10.$

20. $7(a+10)$.

21. $x^2(x+4)$.

Solution:
$x^2(x+4)=x^3+4x^2.$

22. $a^2(a+6)$.

23. $x(x^2+y^2)$.

24. $x^2(x+y)$.

25. $a(2a^2+3a+1)$.

26. $3x(x^2+2x+4)$.

27. $p(2p+r+6s)$.

28. $3y(x+5y+2y^2)$.

29. $2m(m+mn+n)$.

30. $ab(1+2a+b)$.

31. $2a(3a+4b)+a^2(a+3b)$.

32. $x(x^2+ax)+a(a^2+x)$.

33. $4x(x+2y)+y(3x^2+y)$.

Pyramids and Cones

62. Making models of pyramids and cones. We shall now make a study of pyramids and cones (Fig. 77).

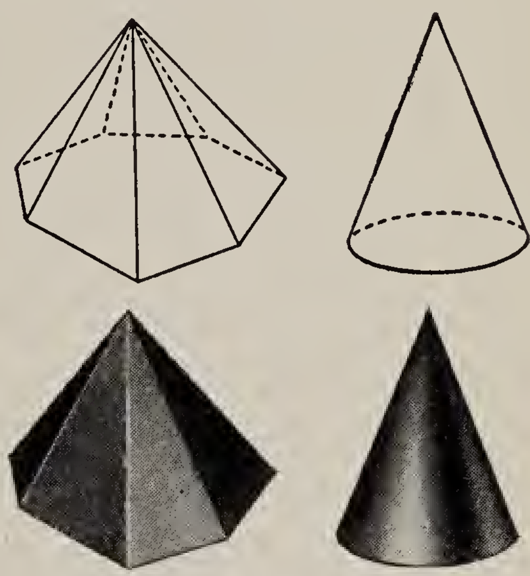


Fig. 77

A model of a pyramid may be made as follows:

Draw a regular polygon, as A (Figs. 78, 79).

On one side of the polygon draw an isosceles triangle, as B.

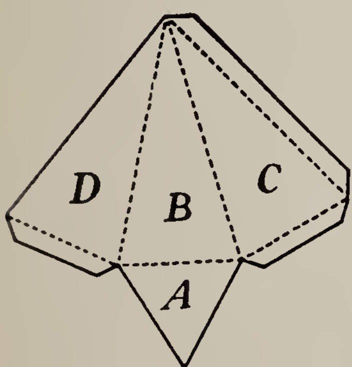


Fig. 78

Construct triangles C, D, etc., congruent to triangle B, making as many triangles as there are sides in the polygon A.

Put on flaps, crease, and fold the paper.

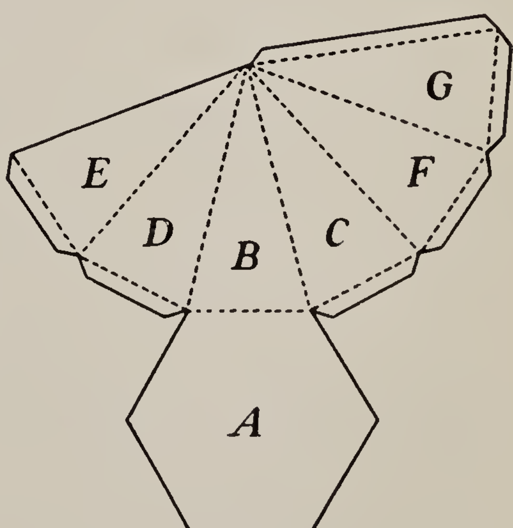


Fig. 79

Using polygon A for the base and the triangles B, C, D, etc., for the lateral surfaces, paste the faces together until the *pyramid* is formed (Figs. 80, 81).

Note that the base of the pyramid (Figs. 80, 81) is a regular polygon and that the vertex is directly above the center of the base. Such pyramids are called *right* or *regular* pyramids.

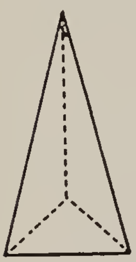


FIG. 80

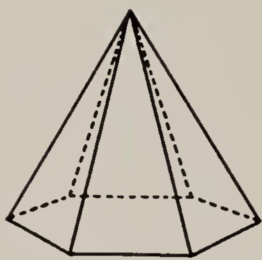


FIG. 81

To make a model of a right circular cone (Fig. 82), draw a circle, as $\odot A$ (Fig. 83). With a convenient, but larger, radius draw a second circle B just touching circle A.

Roll circle A along B, circle B always touching A, and mark off arc CMD equal in length to the circle A.

Draw CB and DB. Put on flap and cut along the solid lines. Fold and paste, using circle A for the base, and the surface $CBDM$ for the curved surface of the cone (Fig. 83).

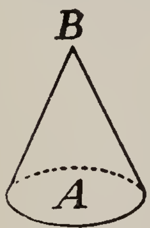


FIG. 82

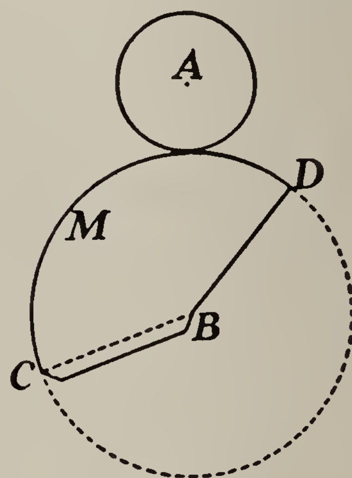


FIG. 83

Make several drawings of the pyramids (Figs. 80, 81) and the cone (Fig. 82).

63. A formula for finding the lateral area of a regular pyramid. Denote the base AB of one of the lateral faces ABC (Fig. 84) of a regular pyramid by b and the altitude by s.

Then the area of $\triangle ACB = \frac{1}{2}bs$.

The altitude CE of triangle ACB is called the *slant height* of the pyramid.

Similarly, the area of $\triangle BCD$, and of each of the other lateral faces, is $\frac{1}{2}bs$.

If the pyramid has four lateral faces the lateral area is $4 \times \frac{1}{2}bs$ which may be changed to the form $\frac{1}{2}(4b)s$. If the pyramid has five faces the lateral area is $\frac{1}{2}(5b)s$. What is the lateral area of a pyramid having six lateral faces?

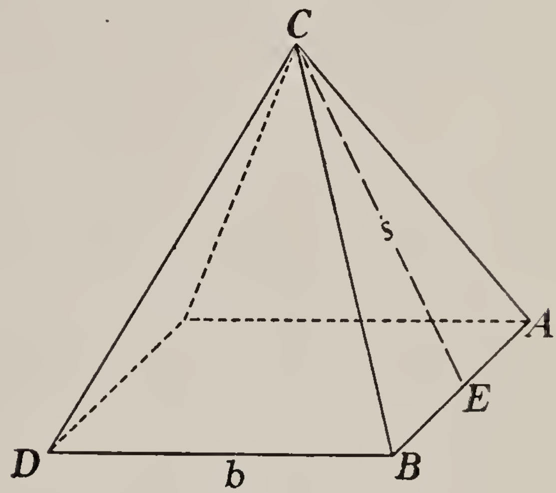


Fig. 84

Notice that $4b$, $5b$, etc., are the *perimeters of the bases* of the pyramids.

We can now state the principle for finding the lateral area:

The lateral area of a right pyramid is equal to one-half the perimeter of the base multiplied by the slant height.

In symbols this may be expressed as follows:

$$\mathbf{L} = \tfrac{1}{2}\mathbf{ps},$$

where L is the number of square units in the lateral surface, p the number of units in the perimeter of the base, and s the number of units in the slant height.

The same formula is used for finding the *lateral area of a cone.* In this case the perimeter of the base is the circumference of a circle, and the formula changes to

$$L = \tfrac{1}{2}(2\pi r)s,$$

which reduces to $\mathbf{L} = \pi\mathbf{rs}$.

Exercises

1. Find the lateral area of the pyramid and cone constructed in §62.

For each of the exercises below make a sketch before solving, as shown in Exercise 2:

2. Find the number of square feet of lumber which was used in constructing the roof of a silo (Fig. 85) whose slant height is 8.5 feet and the diameter of whose base is 15 feet.

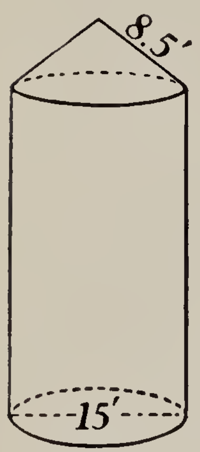


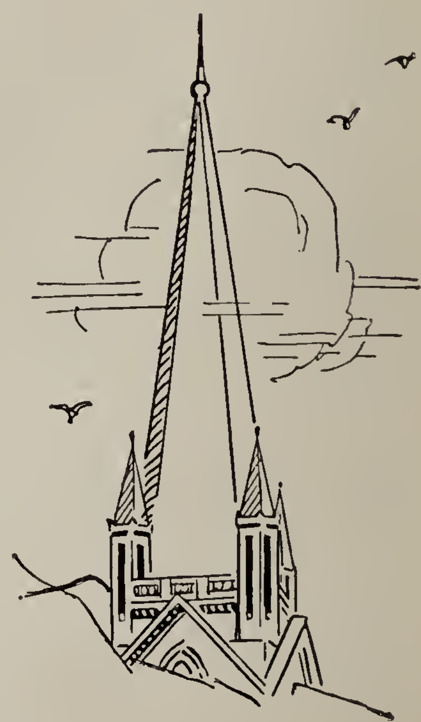
FIG. 85

Solution:	*Computation:*
$L=\pi rs$	3.14
	85
$r=7.5$	1 57
	25 12
$s=8.5$	26.69
	7.5
$\therefore L=(3.14)\ (8.5)\ (7.5)$	13 34
	186 83
$L=200.2$ approximately.	200.17

3. Find to three figures the lateral area L and the total area T of a cone if the slant height is 24 inches and if the radius of the base b is 12 inches.

Suggestion: $T=L+b$.

4. The slant height of a cone is 5.5 feet and the radius of the base is 2.5 feet. Find the lateral area.



5. The base of a right pyramid is an equilateral triangle whose side is 4 feet and whose area is 6.8 square feet. If the slant height is 8 feet, find the area of the lateral surface; of the total surface.

6. The base of a right pyramid is a regular hexagon the side of whose base is 4.2 inches. If the slant height is 5.7 inches, find the lateral area.

7. The slant height of a conical tent is 12 feet. The diameter of the base is 14 feet. Find how many square yards of canvas were needed to make the tent, not allowing for seams and waste.

8. Find the lateral surface of a church spire, the side of whose six-sided base is 10 feet long, if the slant height is 85 feet.

64. Formulas for finding volumes of pyramids and cones. We may work out a formula as follows: Make cardboard models, one of a pyramid and one of a prism having the same height and equal bases (Fig. 86). Fill the pyramid with sand and then pour the sand into the prism (Fig. 87). With well-made models and careful work you will find that it takes three pyramids full of sand to fill the entire prism. In fact, it can be demonstrated by more advanced mathematical methods that the content of the pyramid is exactly one-third that of the prism.

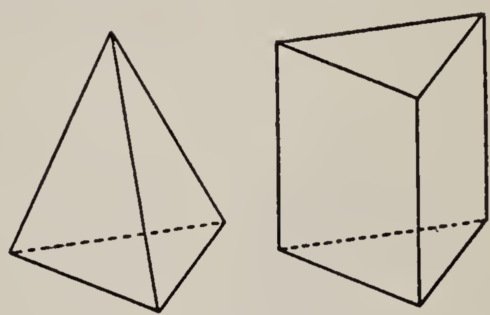


FIG. 86

The same is true for a cone and cylinder (Fig. 88) having equal bases and altitudes. Thus, it appears that (*1*) *the volume of a pyramid is one-third of the volume of a prism having the same base and altitude;* (*2*) *the volume of a cone is one-third of the volume of a cylinder having the same base and altitude.* Since the volume of a prism, or of a cylinder, is equal to the base times the altitude, it follows that *the volume of a pyramid, or of a cone, is one-third of the product of the base times the altitude.*

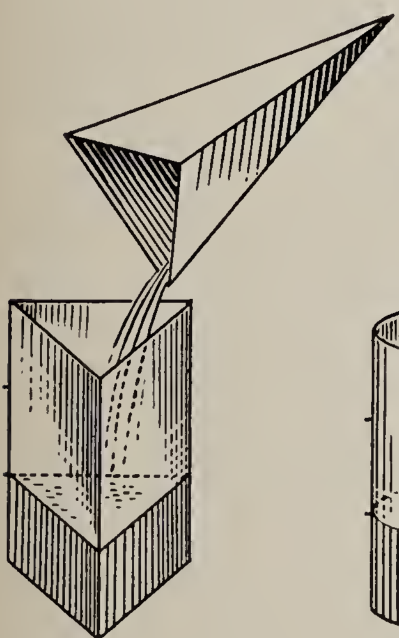


FIG. 87

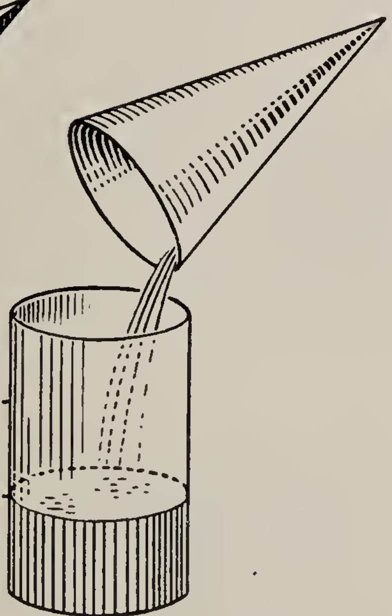


FIG. 88

In symbols we state:

$$V=\tfrac{1}{3}bh, \text{ for the pyramid}$$

$$\text{and } V=\tfrac{1}{3}bh=\frac{\pi r^2h}{3}, \text{ for the cone.}$$

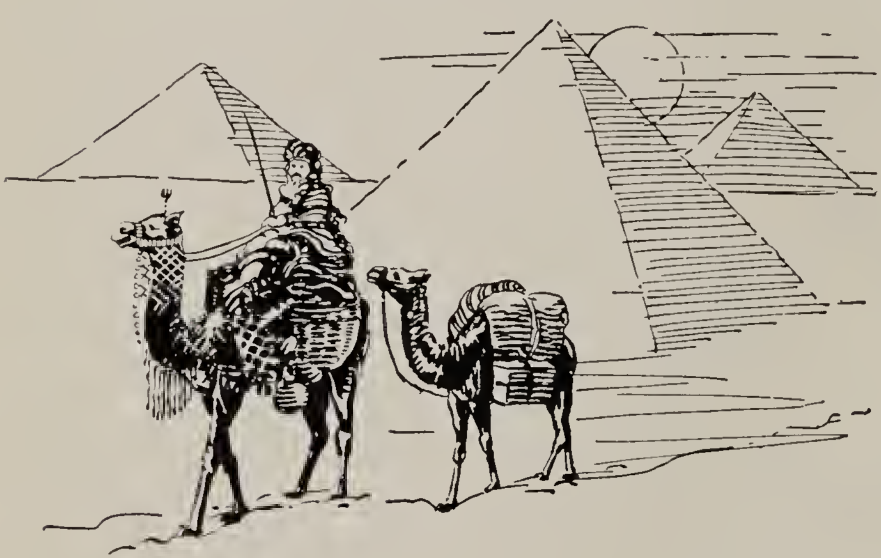
Exercises

1. Find the volume of a cone whose altitude is 8 inches and whose base is 6 inches in diameter.

2. The base of a pyramid is a rectangle 4 inches by 7 inches and the altitude is 12 inches. What is the volume?

3. Find the volume of a square pyramid the side of whose base is 12 inches and whose altitude is 16 inches.

4. A pile of grain has the shape of a cone. The diameter is 8 feet and the height is 3 feet. Find the volume.



5. The great pyramid of Cheops was originally 481 feet high, and the side of the square base was 764 feet long. These dimensions are now 460 feet and 746 feet respectively. How much coating has been removed?

The Sphere

65. What is meant by a sphere. A marble and a ball are illustrations of a sphere. The earth's form is approximately spherical. To be precise, we say that a **sphere**

is a curved closed surface all points of which have the same distance from a fixed point within.

A drawing of a sphere usually represents the visible boundary line $ABCD$ (Fig. 89) and one or more other circles drawn on the surface, as $DEFB$.

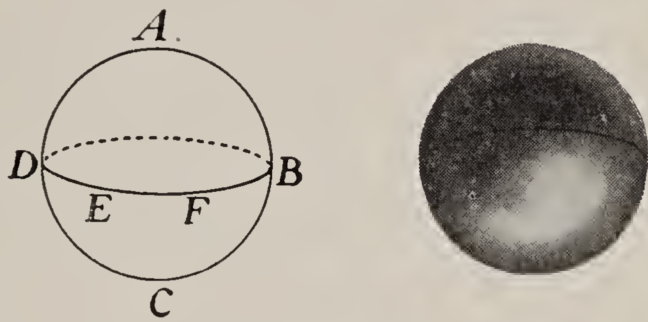


FIG. 89

The formulas for finding the volume and area of the sphere are not so easily derived as those for the other solids studied in this chapter. These formulas are worked out in an advanced course in high-school mathematics.

66. The area of a sphere. It has been proved by methods of geometry that the *area*, S, of a sphere may be found from the formula

$$\mathbf{S = 4\pi r^2}$$

where r is the radius of the sphere.

One way of checking this formula is to wind a heavy cord around the surface of a hemisphere (Fig. 90). Show that this cord covers the surface of the circular flat part of the hemisphere

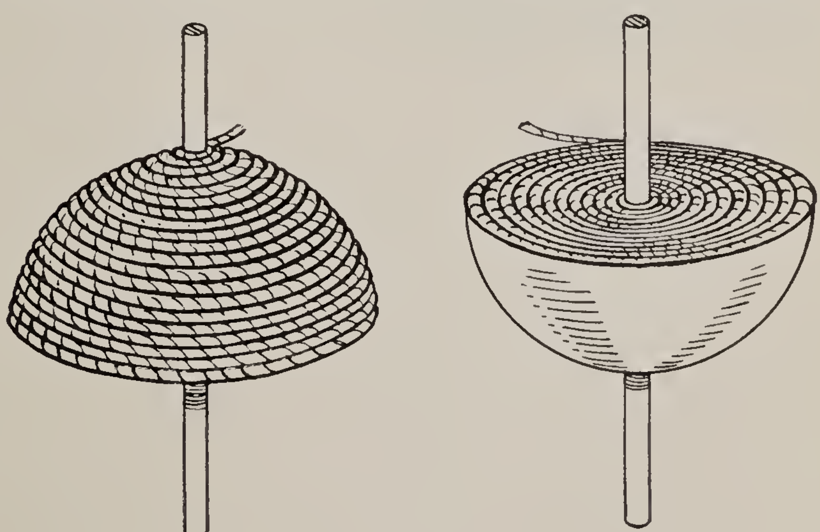


FIG. 90

just twice. Thus the surface of the hemisphere is equal to $2\pi r^2$, and that of the whole spherical surface is $4\pi r^2$.

67. The volume of a sphere. The *volume* of a sphere is found from the formula

$$\mathbf{V} = \tfrac{4}{3}\pi \mathbf{r}^3.$$

Exercises

1. Find the area and volume of a sphere whose radius is 8 inches.

2. Find the weight of a brass ball 4 inches in diameter, assuming that a cubic inch of brass weighs $\frac{3}{10}$ of a pound.

3. Find how many million square miles there are on the surface of the earth. (Radius = 4000 miles approximately.)

4. An iron bar used as a weight in a self-regulating heating plant is $4\frac{1}{2}$ inches in diameter. Find the weight of the ball if iron weighs 480 pounds a cubic foot.

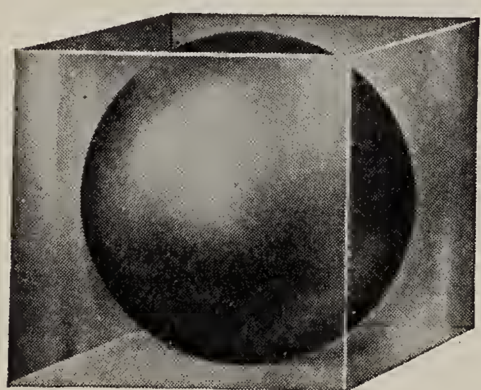


Fig. 91

5. The radius of a sphere is 6 inches. It fits exactly into a cube (Fig. 91). By means of the ratio compare the volume of the sphere with the volume of the cube.

Find the value of each of the polynomials in Exercises 6 and 7:

6. $x^3 + 3x^2y + 3xy^2 + y^3$ when $x = 2$, $y = 1.5$.

7. $2a^3 + 4a^2b + 3ab^2 + b^3$ when $a = 3$, $b = 2$.

In Exercises 8 to 13 multiply as indicated (§13) and check results by substituting values:

8. $y(3x^2 + 3xy + y^2)$.

9. $(3x^2 + 2x + 1)\ (x + 4)$.

10. $(4a^2 + 10a + 1)\ (2a + 6)$.

11. $(a + b + c)\ (m + n + t)$.

12. $(a^2 + ab + b^2)\ (a + b)$.

13. $(2 + x + x^2)\ (x + 4)$.

Formulas and Tables

68. Summary of formulas. The following table summarizes important formulas studied in this chapter:

Area and Volume

	Area	*Volume*
Cube	$L=4e^2$, $T=6e^2$	$v=e^3$
Prism	$L=ep$	$v=bh$
Cylinder	$L=2\pi rh$	$v=\pi r^2h$
Pyramid	$L=\frac{1}{2}sp$	$v=\frac{1}{3}bh$
Cone.......	$L=\pi rs$	$v=\frac{1}{3}bh$ $=\frac{\pi r^2h}{3}$
Sphere......	$S=4\pi r^2$	$v=\frac{4}{3}\pi r^3$

Table of Cubic Measure

1728 cubic inches	= 1 cubic foot
7.48 gallons	= 1 cubic foot
27 cubic feet	= 1 cubic yard

69. What every pupil should be able to do. When you have completed this chapter, you should be able to do the following:

1. To give the names of the common solids.
2. To make good drawings of each solid.
3. To state the formulas for finding areas and volumes.
4. To state the meaning of such algebraic expressions as x^2, x^3, $3x^2$, $5x^3$, and of combinations made up into polynomials such as $3x^3-2x^2+4x-1$, or $\frac{a^3+2b+c}{3c-ab}$;

5. To find the values of polynomials for given values of the literal numbers.

6. To find products of given polynomials.

7. To solve simple cubic equations like $2x^3=128$.

8. To solve with ease simple verbal problems of finding areas and volumes of solids.

70. Typical exercises and problems. The following problems and exercises are typical of the work of this chapter. Every pupil should be able to work them, and others like them.

1. Make a drawing of each of the following solids:

cube	hexagonal prism
rectangular solid	cylinder
triangular prism	circular cone
triangular pyramid	sphere
square pyramid	

2. State the formulas for finding the areas of the solids which were studied in Chapter III.

3. State the formulas for finding the volumes of these solids.

4. Find the value of the diagonal of a cube whose edge is 3.5 inches.

5. Find the exact value of x^3+3x^2+3x-4 when $x=2.6$.

6. Find the exact value of $\dfrac{ab^2+a^2b+c}{a+b+c}$ when $a=1.4$, $b=3.1$, $c=.6$.

7. Simplify: $3x \cdot 2x^2$; $x \cdot 8x \cdot \frac{1}{4}x$; $(2a)^3$; $(2m)^2+(3n)^3$.

8. Multiply as indicated: $3a(x^2+2a+8)$; $2x(x+5y+8y^2)$.

9. Multiply $(3a^2+5a+1)$ by $(a+2)$.

10. Solve the equation $2x^3=128$.

11. Find the number of board feet in a beam 3 inches by 4 inches and 14 feet long.

12. Find to three figures the volume of a right prism whose altitude is 13.4 inches and the area of whose base is 65.3 square inches.

13. Find the volume of a cylindrical tank 50 feet in diameter and 36 feet high.

14. The slant height of a cone is 5.5 feet and the radius of the base is 2.5 feet. Find the lateral area.

15. Find the area and the volume of a sphere whose radius is 4 inches.

16. Write a paper on one of the following topics:

a. The use of algebra in problems of areas and volumes of solids.

b. Mathematics used in various occupations, as farming, contracting, building.

CHAPTER IV

THE MEANING OF POSITIVE AND NEGATIVE NUMBERS

Directed Numbers

71. Various types of numbers. In the very earliest times people knew only *whole* numbers. They used them in counting or in comparing a magnitude with a standard unit very much as we do to-day when we find the distance between two trees or other objects by comparing it with the length of a foot.

With the development of the human race other kinds of numbers have appeared, *e.g.*, the *fractions*, as $\frac{2}{3}$ and $\frac{3}{4}$, and numbers like $\sqrt{2}$ and $\sqrt{5}$. At first, these new numbers were not understood and for this reason they were at times called *artificial numbers*. Similarly, in the study of geometry and of the physical sciences, and in everyday affairs there arose a need for other artificial numbers.

In this chapter we shall learn about a new kind of number. One difference between the new numbers and the numbers that you worked with in arithmetic is that $+$ or $-$ sign is prefixed to the figure, as -2, or $+8$. When such numbers first appeared in mathematical work, mathematicians rejected them because they were unable to attach any meaning to them. They called them *fictitious* or *absurd* numbers.

We shall call them *signed numbers.* You will find it interesting to read more about the development of the number system in books on the history of mathematics.

When the meaning of signed numbers became clear, they were readily accepted. To-day they are employed for many purposes. We shall see that signed numbers are used in temperature readings, in designating opposite directions, and in business. We shall study them until we understand them thoroughly.

72. Signed numbers are used in thermometer readings. On a winter day the weather report published in the newspapers gave the following table stating the temperature for 24 hours:

Maximum	2 P.M.	7
Minimum	2 A.M.	−5

3 A.M.	−4	11 A.M.	5	7 P.M.	6
4 A.M.	−3	Noon	6	8 P.M.	5
5 A.M.	−2	1 P.M.	6	9 P.M.	4
6 A.M.	−1	2 P.M.	7	10 P.M.	1
7 A.M.	2	3 P.M.	6	11 P.M.	0
8 A.M.	4	4 P.M.	6	Midnight	−3
9 A.M.	4	5 P.M.	5	1 A.M.	−4
10 A.M.	6	6 P.M.	6	2 A.M.	−5

The numbers in the right-hand columns in this table refer to the thermometer scale (Fig. 92). A certain point on this scale is designated as the *zero point.* The numbers −5, −4, −3, in the table mean, respectively, 5° below zero, 4° below zero, and 3° below zero. Thus degrees *below* zero are denoted by numbers prefixed by

a *minus sign* (–). The numbers 2, 4, 6, etc., in the table mean: 2° above zero, 4° above zero, and 6° above zero. Sometimes readings *above* zero are denoted by numbers prefixed by a *plus sign* (+). Accordingly, the readings +6°, +4°, +2° mean that the temperature is 6° above zero, 4° above zero, 2° above zero.

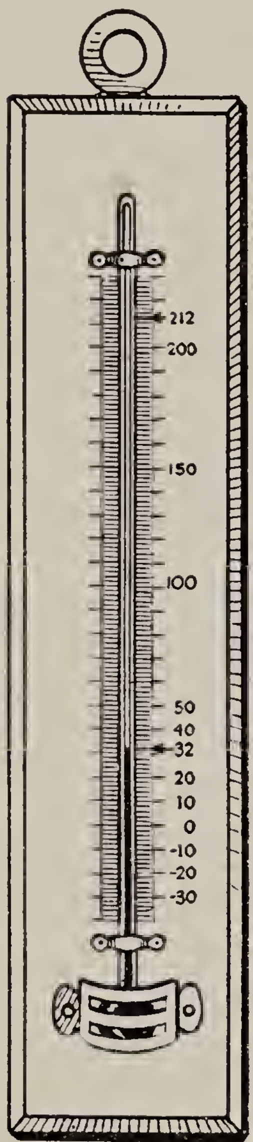


Fig. 92

Exercises

1. State the meaning of each of the following temperature readings: +60°, –3°, –8°, 0°, +4°, –10°, +88°.

2. If the top of the mercury column in a thermometer is at 5°above zero and the temperature then rises 4°, express the final reading by means of the + or – sign. If the temperature then falls 11°, state the final reading.

3. State the final readings in the table below:

First reading	8°	6°	2°	3°	–8°	12°	–3°	0°	–2°
Change	rise of 2°	fall of 10°	fall of 6°	rise of 4°	fall of 6°	fall of 12°	fall of 8°	rise of 10°	rise 8°
Final readings									

4. Represent the readings in the table on page 103 graphically as follows:

On the line OX (Fig. 93) lay off distances representing the hours. Then at each hour point lay off, at right angles to OX, the corresponding temperature reading. Connect the points thus located, forming the *temperature line.*

Examine the graph and answer the following questions: When was the temperature highest? When was it lowest? When was the change in temperature greatest? What was the maximum temperature? The minimum?

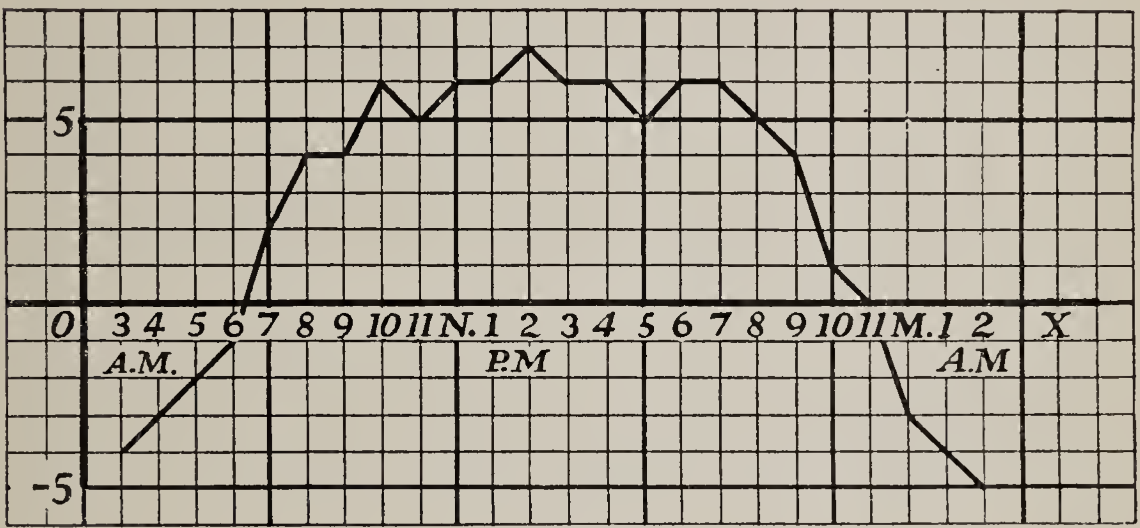
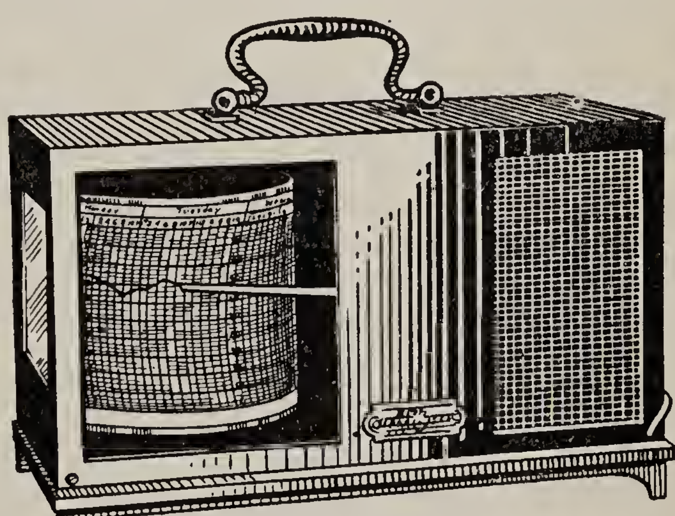


Fig. 93

The graph registers the variations of the top of the mercurial column when hourly readings are taken. To get a continuous record, a self-recording thermometer is used (Fig. 94). The instrument is called *thermograph* because it represents temperature graphically. A thermometer is attached to a pen which rises and falls as the temperature changes, making a continuous line on the cylinder, which is turned horizontally by a clock. The days and hours are marked by vertical lines and the degrees by horizontal lines.

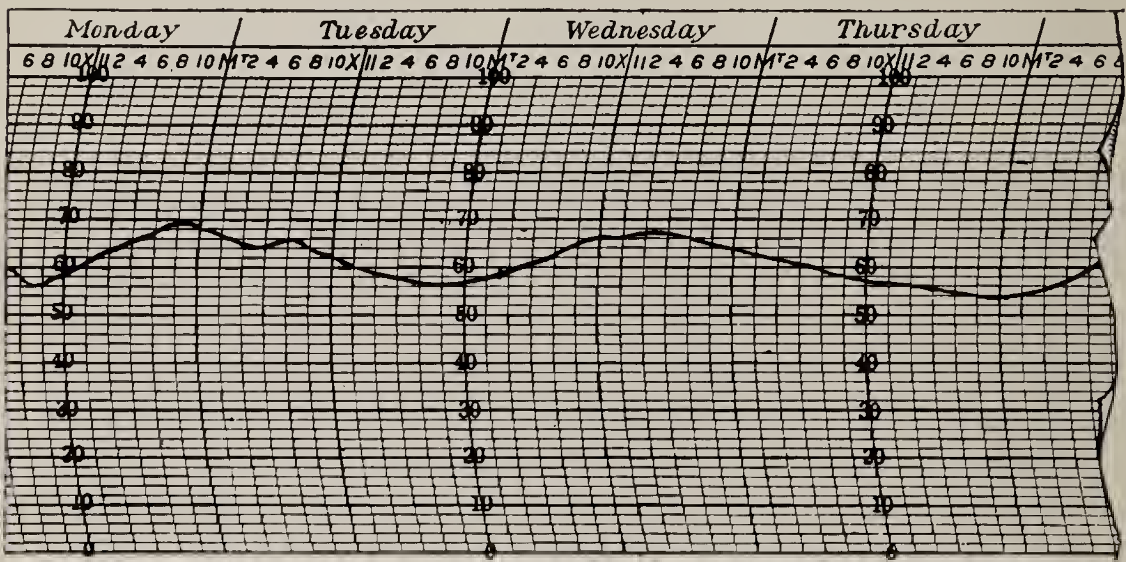


Courtesy Taylor Instruments Co.

Fig. 94. A Thermograph.

Thermographs are used at Weather Bureau stations, in laboratories, and in conservatories. When removed from the cylinder

of the thermograph the temperature graph appears as shown in Fig. 95. It may be used to determine the *average* temperature for the month.



Courtesy Tayl r Instruments Co.

FIG. 95

5. For a certain day the hourly temperature readings beginning at 8:00 A.M. were as follows: 10°, 12°, 14°, 18°, 20°, 22°, 21°, 18°, 14°, 8°, −2°, −4°. Make the graph and tell what it shows.

6. Represent graphically the following daily *average* temperatures for one week: +10°, 0°, −8°, −4°, +6°, +14°, +15°.

7. Make a graph for the following table:

Time	Midnight	1 A.M.	2	3	4	5	6	7	8	9
Temperature	−10°	−12°	−15°	−13°	−10°	−9°	−8°	−6°	−3°	0°

Time	10	11	Noon	1	2	3	4	5	6	
Temperature	2°	5°	7°	7°	9°	10°	8°	8°	7°	

73. Meaning of the signs in directed numbers. In arithmetic the + and − signs are used to indicate addition and subtraction. We have just seen that

they may be used to denote opposite directions from a fixed point, *e.g.*, from the zero point on the thermometer scale. This explains why numbers preceded by a sign, + or −, are called *directed* numbers. By the introduction of directed numbers into the number system the power and scope of mathematics was greatly increased.

74. Directed numbers are used to denote above or below water level. Heights of mountains and depths of lakes are measured in opposite directions from the surface of the water. The height of point A (Fig. 96) is $+8'$, *i.e.*, 8 feet above water level. The position of point Q is $-5'$, *i.e.*, 5 feet below.

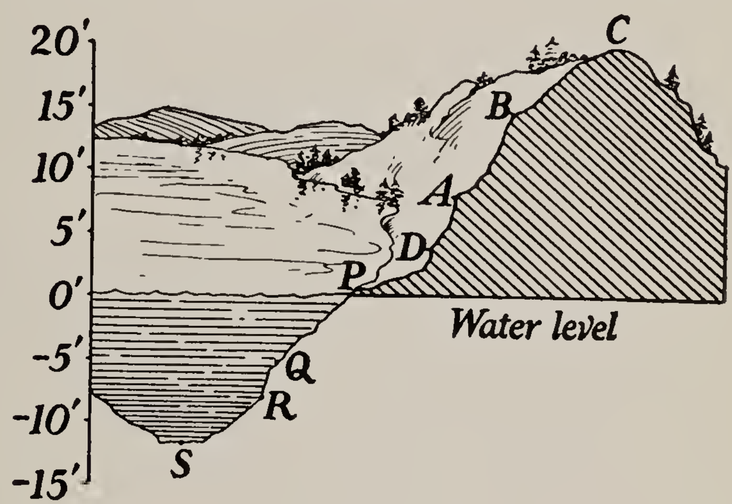


Fig. 96

Exercises

1. By means of + and − signs state the position (height and direction) of points B, C, D, P, R, S (Fig. 96).

2. In a certain survey heights and depths were measured at intervals of 10 feet and recorded as in the table below. Make a diagram like that of Fig. 96 representing the facts contained in the table.

Horizontal distances	0	10	20	30	40	50	60	70	80	90	100	110	120	130
Heights or depths	−6	−12	−22	−20	−25	−20	−7	−2	+4	+5	+7	+6	+3	0

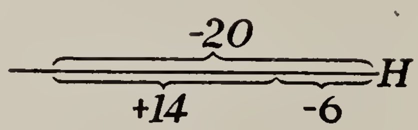
75. Directed line-segments. The location of a place may be denoted by means of a $+$ or $-$ sign. Thus directions to the north are usually considered $+$ and directions to the south $-$.

Exercises

In the following exercises directions are to be denoted by $+$ and $-$ signs, as shown in Exercise 1.

1. A boy riding a bicycle starts from home, rides 20 miles west, *i.e.*, in the negative direction, turns, and rides 14 miles in the opposite direction. How far is he from home?

Solution:



$$\therefore -20+14=-6.$$

2. If a man travels 50 miles north one day and 76 miles south the next day, how far is he from the starting point?

Arrange the solution as shown in Exercise 1.

3. An elevator goes up 128 feet (eight floors) and then down 32 feet (two floors). How far is it from the first floor?

4. An elevator goes up 56 feet and down 70 feet. How far is it from the starting point?

76. Directed angles. The rotation of a radius about point B (Fig. 97) from the position of BA to that of BC forms angle ABC. Turning the radius the same amount, a, from BA to BC' forms angle ABC'. The two angles have the same numerical measure but the directions of turning are opposite. We shall denote

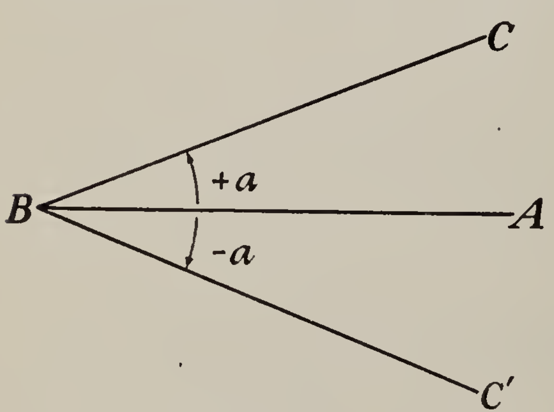


Fig. 97

clockwise rotation by the $-$ sign and *counterclockwise* rotation by the $+$ sign. Hence, $\angle ABC' = -a$, and $\angle ABC = +a$.

In geography, navigation, and astronomy directed angles are used to locate objects and places; *e.g.*, latitudes are $+$ or $-$ according as they are north or south of the equator.

Exercises

Use plus or minus signs in stating the results in the exercises below:

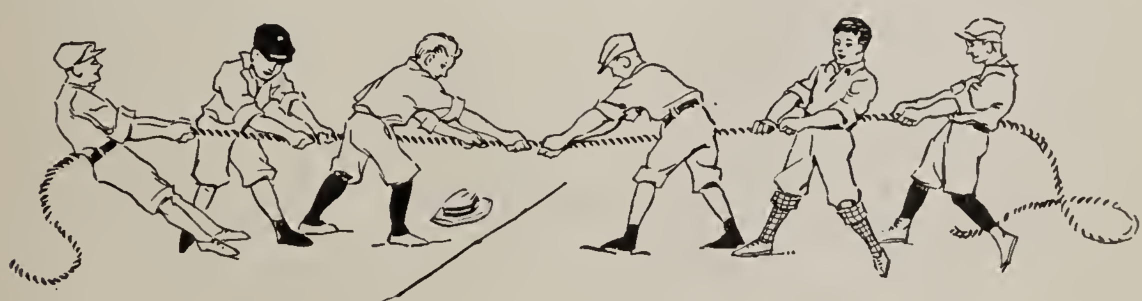
1. Draw the following angles, starting in each case with the initial line taken in the direction from left to right, as BA (Fig. 97): $+25°$, $-40°$, $-120°$, $180°$, $-220°$. Indicate in the drawing both size and direction.

2. What sign should be prefixed to the latitude of each of the following cities: New York City, St. Louis, Cape Town, Boston, Rio de Janeiro, Galveston?

3. In what latitude is a traveler who starts in latitude $-3°$ and travels north $7°$?

4. Find the latitude and longitude of your home city and express each with a signed number.

77. Directed forces. The opposing forces in a tug-of-war, the weight of a stone and the upward pull of a



balloon, the rate of a current and the rate of rowing up-stream are examples of forces acting in opposite directions.

Exercises

Answer the following questions, using the + and − signs to denote directions:

1. If the weight of a stone is denoted by the − sign, how should we denote the weight (upward pull) of a balloon?

2. If a stone weighing 5 ounces is attached to a balloon pulling upward with a force of 7 ounces, what is the magnitude and direction of the combined weight?

3. A boy can row in still water at the rate of 4.5 miles an hour. How fast can he go upstream against a current flowing at the rate of 2 miles an hour? How fast can he go downstream?

4. If the rate of a current is x miles per hour and the rate of rowing y miles, how fast can a man go upstream? Downstream?

5. A carrier pigeon can fly at a rate of 55 miles an hour. How fast can it fly against a wind blowing at the rate of 35 miles an hour?

6. In a tug-of-war one side pulls to the north with a force of 320 pounds, the other to the south with a force of 312 pounds. Express these forces and the resulting force with + and − signs.

78. Use of signed numbers in business. The following is part of a report of the New York stock transactions, taken from a newspaper. The plus and minus signs in the last column are used to indicate the daily rise or fall of stocks.

NEW YORK STOCK TRANSACTIONS

Descriptions	*Sales*	*High*	*Low*	*Close*	*Change*
Adams Exp.	100	$22\frac{1}{2}$	$22\frac{1}{2}$	$22\frac{1}{2}$	$+\frac{1}{2}$
Ajax Rubber	900	25	$24\frac{3}{4}$	$24\frac{3}{4}$	$-\frac{1}{4}$
Am. Sugar.	2,600	$90\frac{1}{2}$	$88\frac{3}{4}$	89	$+1$
Crucible Steel	15,500	$74\frac{1}{4}$	$71\frac{1}{2}$	$71\frac{7}{8}$	$-3\frac{5}{8}$
Gen. Motors	7,500	$15\frac{3}{8}$	$13\frac{3}{8}$	$13\frac{5}{8}$	$+\frac{1}{8}$
Peoples Gas.	500	$33\frac{1}{2}$	$32\frac{1}{8}$	$32\frac{3}{4}$	$-\frac{3}{4}$
Western Union.	300	$82\frac{1}{4}$	$81\frac{1}{2}$	$81\frac{1}{2}$	. . . .

From the last column tell the rise or fall of each stock.

EXERCISES

1. State the opposites of the following terms: gain, increase, deposit, possession, export, asset, after Christ, north, forward, fall, above zero. If either of two opposites is denoted by a + sign the other should have a minus sign.

2. A man is in debt \$300 and borrows \$450. He then receives \$1000. How much is he worth?

Solution: $-300 - 450 = -750$
$-750 + 1000 = +250.$

In the following exercises arrange the solutions as in Exercise 1:

3. A man's property is worth \$8500 and his debts amount to \$3600. Express his financial standing by means of a signed number.

4. A man's account book contains the following items: salary \$416, rent \$85, food \$75, insurance \$30, interest on bonds \$22. Denote these items with plus or minus signs, and determine his financial standing.

5. The assets of a company are \$38,328 and the liabilities are \$35,220. What is the financial standing?

6. A merchant gains \$8115 one year and loses \$1876 the next. Find his net gain or loss for the two years.

7. A man's monthly bank statement reads as follows:

Date	*Checks*		*Date*	*Deposits*	*Date*	*Balance*
	Balance brought forward			. . \$124.07		
Oct. 1	\$35.00–	\$5.00–	Oct. 1	\$329.16	Oct. 1	\$413.23
Oct. 2	10.00–		Oct. 2	67.50	Oct. 2	470.73
Oct. 4	10.00–	30.00–	Oct. 4	80.25	Oct. 4	510.98
Oct. 5	18.50–	19.25–			Oct. 5	473.23
Oct. 6	112.66–	50.00–			Oct. 6	310.57
Oct. 8	40.00–	5.00–			Oct. 8	237.07
Oct. 9	45.00–	28.50–	Oct. 9	117.50	Oct. 9	309.57

The last amount in the last column to the right is the balance on Oct. 9. For each date verify the correctness of the statement. Note that instead of prefixing the – sign the bank places it after the number, + signs in deposits are omitted but understood.

Positive and Negative Numbers

79. The number scale. A number preceded by a + sign is called a **positive** number, and one preceded by a – sign is a **negative** number.

We have read above (§71) that in early times negative numbers were not approved because people were not able to understand them. The first to see a real meaning and use for signed numbers were the Hindus. Aryabhatta (born 476 A.D.) recognized their value in making distinction between assets and liabilities. European mathematicians were exceedingly slow to accept negative numbers. Some were willing to admit them when they arose in problems in which they could be given a meaning such as debts, but positive

and negative numbers were not fully accepted until Descartes (1596–1650) used them systematically in his famous geometry. He represented them graphically by means of line segments.

DESCARTES

René Descartes was born at La Haye, near Tours, March 31, 1596, and died at Stockholm, February 11, 1650.

In the year 1637 he wrote a book, *Discourse on Methods*, which contained an appendix on geometry. He showed how to study geometrical figures by means of algebraic equations. This added much to his fame and mathematical reputation. Read more about the life and work of Descartes in Ball's *History of Mathematics*, pp. 268–272.

It must be remembered that positive and negative numbers are composed of two parts, the sign and the arithmetical value. The latter is at times called the *numerical* or *absolute* value. The + sign is frequently omitted. A number without a sign is, therefore, positive.

To represent positive and negative numbers graphically, arrange them along a straight line (Fig. 98), the *positive* numbers to the *right* of the zero and the *negative* numbers to the *left*. This is the **number scale**. Any number in this scale is con-

Fig. 98

sidered as *less* than all numbers to the right and as *greater* than all numbers to the left. At whatever point

of the scale we may start, if we pass to the right the numbers are increasing, and if we pass to the left they are decreasing. Thus -4 is *greater* than -6, although 4 is numerically *less* than 6.

Exercises

1. Name the greater number in each of the following number pairs: $0, +5$; $-3, 0$; $-2, -5$; $+6, -4$; $-10, +10$.

2. By how much is -8 less than -3? Verify your answer with the number scale.

3. Locate the following numbers on the scale: $+6$, -9, $\frac{2}{3}$, $-\frac{3}{4}$, $2\frac{1}{2}$, $+3\frac{1}{3}$, 0, -6.3.

4. At a certain hour the temperature was $+3°$. Two hours later it was $-2°$. What was the drop (difference) in temperature? Verify your answer with the number scale.

5. Augustus lived from the year -63 to the year $+14$. How old was he when he died?

80. What every pupil should know and be able to do. Chapter IV has shown the meaning and uses of positive and negative numbers. The pupil should now understand the uses of signed numbers with the thermometer scale, angles, segments, forces, business terms and the number scale.

81. Typical exercises and problems. The following problems and exercises review the essential facts taught in Chapter IV. Every pupil should be able to do them.

1. Name uses for positive and negative numbers other than those mentioned in Exercise 1, § 78.

2. Represent the following table graphically:

Time	4 P.M.	5	6	7	8	9	10	11	Midnight	1 A.M.	2	3
Temperature	$+8°$	$+8°$	$+6°$	$+4°$	$+3°$	$0°$	$-1°$	$-2°$	$-2°$	$-3°$	$-4°$	$-4°$

For each of Exercises 3 to 6 write an equation stating the facts of the problem and the result:

3. An automobile travels 22 miles east and then 45 miles west. How far is it from the starting point?

4. A ship starts in latitude 5° north and travels 12° south. What is its latitude?

5. The rate of a current is $2\frac{1}{2}$ miles an hour. If a boy can row in still water at a rate of 3.5 miles an hour, how fast can he go upstream?

6. A man's bank account is $400, but he owes $250. What is his financial standing?

7. Write a paper on one of the following topics:

a. The development of the number system, including positive and negative numbers.

b. The meaning and uses of positive and negative numbers.

CHAPTER V

THE OPERATIONS WITH POSITIVE AND NEGATIVE NUMBERS

Addition

82. Why we should know how to work with positive and negative numbers. In arithmetic, the statement \$6 − \$4 has meaning but \$4 − \$6 has no meaning. On the scale of numbers (Fig. 99), "6 − 4" means that starting from zero we pass 6 units to the right and from that point 4 units to the left. This takes us to a point 2 units to the right of zero.

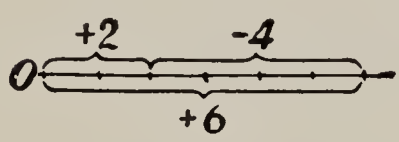


Fig. 99

Briefly we may say $+6-4=+2$.

By the same process we find that $4-6$ means that we pass 4 units to the right of zero (Fig. 100) and then 6 units to the left. The result is -2.

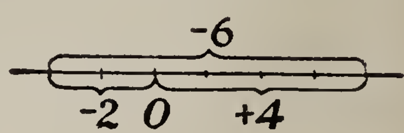


Fig. 100

Briefly, $+4-6=-2$.

With arithmetical numbers we are able to solve the following problem: If a number is increased by 4 the result is 6. Find the number.

Solution: Let x be the number.

$$\begin{aligned} \text{Then } x+4 &= 6 \\ 4 &= 4 \\ \hline \therefore x &= 6-4, \\ \text{or } x &= 2. \end{aligned}$$

However, we need to understand positive and negative numbers to solve the following problem, which is similar to the one above: If a number is increased by 6 the result is 4. Find the number.

Solution: Let x be the number.

$$\begin{array}{rl} \text{Then } x+6 & =4 \\ 6 & =6 \\ \hline \therefore x & =4-6, \\ \text{or } x & =-2. \end{array}$$

Verify the result on the number scale.

In future work we are to meet many other problems whose solution requires knowledge of signed numbers. The interpretations and uses of positive and negative numbers that were shown in Chapter IV have helped us to *understand* signed numbers as readily as arithmetical numbers. However, we are not prepared to take up problems involving signed numbers until we know *how to work with them, i.e.,* how to add, subtract, multiply, and divide them. In this chapter we are going to learn how to perform these operations.

83. How to add signed numbers graphically. To add $(+5)$ and $(+3)$ lay off on the number scale (Fig. 101) first $(+5)$ in the positive direction OA, and then $(+3)$ in the positive direction. *The distance and direction* $(+8)$ *from the starting point to the stopping point is the required sum.*

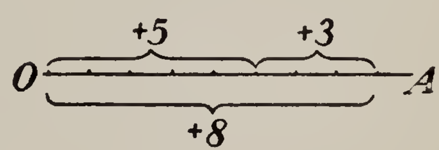


Fig. 101

Thus we have the following equation:

$$(+5)+(+3)=+8.$$

This equation states briefly that 5 to the right and then 3 more to the right gives 8 to the right or $+8$.

Similarly, $(+5)+(-3)=(+2)$. For, $(+5)$ is laid off first (Fig. 102) in the positive direction to B. Then (-3) is laid off from B in the negative direction BO, and the direction and distance from the starting point O to the stopping point C is $+2$.

We may say that the statement $(+5)+(-3)=(+2)$ means that 5 to the right and then 3 to the left gives 2 to the right.

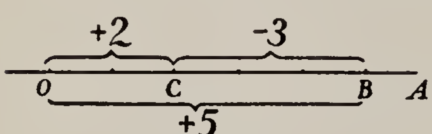


FIG. 102

The two preceding examples show that two signed numbers may be added *graphically* as follows:

Let a *and* b *be two numbers, positive or negative. To find the sum of* a *and* b *lay off, on the number scale, first* a *in its own direction and then* b *in its own direction. The distance and direction from the starting point to the stopping point is the required sum.*

EXERCISES

Find the sums in Exercises 1, 2, and 3 graphically:

1. $(-8)+(+3)$.

Solution: From O (Fig. 103) lay off -8 in its own direction to A. From A lay off $+3$ in its own direction to B. Then OB, or -5, is the required sum.

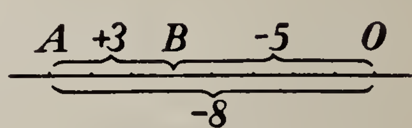


FIG. 103

Hence $(-8)+(+3)=(-5)$.

2. $(+10)+(-4)$; $(-6)+(-3)$; $(-7)+(8)$; $12+(+3)$; $-10+(6)$.

3. $6+(-6)$; $-4+4$; $5+(-5)$.

4. Show by means of several examples, as in Exercise 3, that the sum of two numbers having the same arithmetical value and opposite signs, as $+a$ and $-a$, is always zero.

5. What must be added to each of the following to make the resulting sum zero: $+4$; -6; $+3$; -1?

6. Add the following *mentally*, *i.e.*, use the number scale without actually drawing the figure: $+16+(-3)$.

Solution: Pass 16 to the right and from there 3 to the left. The result is 13 to the right, or $+13$.

$\therefore +16+(-3)=+13.$

Similarly, add: $+3+(-6)$; $+4+(+8)$; $-2+(+1)$; $-8+(+12)$; $-5+(-7)$.

7. Add mentally:

$-8+(+4)+(-3)$; $+3+(-2)+(+4)$; $-6+8+(-7)$; $4+8-4$; $6+(-6)+2$.

Solve the equations in Exercises 8 to 18 by adding the same number to both members:

8. $x-8=-2.$

Solution:

Adding 8 to both members we have

$$\begin{array}{r} x-8=-2 \\ 8=8 \\ \hline x-8+8=-2+8 \end{array}$$

$$\text{or } x+8-8=-2+8$$

$$\text{or } \qquad x=6.$$

The solution may now be written in the following simple form:

$$\begin{array}{rcr} x-8 & = & -2 \\ 8 & = & 8 \\ \hline x & = & 6 \end{array}$$

9. $8-x=-6+4x.$

Solution: $8-x=-6+4x$

Adding x, $8=-6+5x$

Adding 6, $14=5x$

$\therefore x=\frac{14}{5}$

or $x=2\frac{4}{5}.$

10. $a-9=-3$. **13.** $4x-8=6-3x$. **16.** $6t-10=8-5t$.

11. $y-4=9$. **14.** $3a-2=-7a+5$. **17.** $10m-4=2m+7$.

12. $3x-10=x-6$. **15.** $5x-7=2-3x$. **18.** $5r-4=9r-7$.

19. Add mentally the lower to the upper:

$+16$	$+8$	$+6$	-4	-3	-7	2
$+\ 4$	-3	-9	-6	$+5$	3	-2

20. Add mentally the lower number to the upper:

$-3x$	$-6a$	$-6y$	$+12m$	$-5(a+b)$	$3(x+y)$
$-2x$	$-\ a$	$+6y$	$+\ 2m$	$2(a+b)$	$-7(x+y)$

84. Combining terms. The sum of two *dissimilar* numbers, as a and b, is $a+b$. If two numbers are *similar*, as $6x$ and $2x$, the sum is $6x+2x$, or $8x$. The $8x$ is said to be obtained by *combining*, or *collecting*, the terms $6x$ and $2x$. Thus, similar terms can be combined, but dissimilar terms cannot be combined.

Exercises

In the following combine similar terms:

1. $16a+(4b)+(-3a)+(-b)$.

Solution: $16a+(4b)+(-3a)+(-b)$
$=16a-3a+4b-b$, by changing the order of the terms.
$=13a+3b$, by combining similar terms.

2. $10+3+(-4)+6-3$.

3. $12+(7)+(-3)+2$.

4. $5+(-2)+6+(-3)$.

5. $7x+2x+5y+(-3y)$.

6. $8a+(-b)+(-3a)+(-5b)$.

Solve the following equations:

7. $-6x+15=3x-30.$

Solution: $-6x+15=3x-30$

Adding $6x$, $15=9x-30$

Adding 30, $45=9x$

Dividing by 9, $5=x$.

8. $x-3=5.$ **11.** $5x-12=-7.$ **14.** $4x-2=x+31.$

9. $2x+4=14.$ **12.** $3x-20=-2.$ **15.** $-x-15=-18+2x.$

10. $7x-8=20.$ **13.** $2x-3=18-x.$ **16.** $10-5x=-6+3x.$

85. A rule for adding numbers. We have learned to add numbers by means of the number scale. We have also seen that we may add numbers more rapidly by making a mental picture of the graphical addition, without actually making the drawing. A third way is to use a rule in adding numbers. The following four problems in addition illustrate this rule.

Add the lower number to the upper:

$$\begin{array}{r}+7\\+5\\\hline+12\end{array}\qquad\begin{array}{r}-7\\-5\\\hline-12\end{array}\qquad\begin{array}{r}+7\\-5\\\hline+2\end{array}\qquad\begin{array}{r}-7\\+5\\\hline-2.\end{array}$$

Note (1) that in the *first two* problems the numbers to be added have *like* signs; (2) that the arithmetical value of the sum may be found by *adding* the arithmetical values of the two numbers; (3) that the sign of the sum is the *same* as the sign common to the given numbers. The three facts are stated in the form of a rule as follows:

To add two or more algebraic numbers having like signs, add the arithmetical values and prefix the common sign to the sum.

In the *third and fourth* problems above, note (1) that the two given numbers to be added have *unlike* signs; (2) that the arithmetical value of the sum is equal to the *difference of the* arithmetical values of the numbers; and (3) that the sign prefixed is the sign of the number having the *greater* arithmetical value. This is expressed by the following rule:

To add a positive and a negative number, find the difference of the numerical values of the numbers, and prefix to it the sign of the numerically greater number.

Exercises

1. Find the following sums by rule and then verify each by graphical addition:

$+8+(+3)$; $+9+(-3)$; $7+(+5)$; $-5+2$; $-6+(-4)$; $+6+(-3)$; $-4+(+6)$; $-2+4$.

2. Show that $6+(-2)=6-2$; $8+(-5)=8-5$.

3. Collect the terms in each of the following polynomials: $6-2+(-8)$; $-8+(-3)-10$; $+3+(+2)+(+8)$; $-6+(-4)+(-1)$.

4. Collect terms in the following:

$3x-2x+6x+4x-x$.

Solution: $3x-2x+6x+4x-x$

$=3x+6x+4x-2x-x$, by changing the order of the terms

$=13x-3x$, by combining terms

$=10x$.

5. $-8x+17x+4x-9x$.

6. $14x-10x-3x+7x$.

7. $-3a-12a-5a+20a$.

8. $12.5b-9.5b-7.3b+b$.

9. $.04m-.03m+.16m+.23m$.

10. $\frac{2}{3}x+\frac{1}{4}x-\frac{3}{4}x+\frac{1}{3}x$.

11. $7a^2-3b+6b-14a^2$.

Solution: $7a^2-3b+6b-14a^2=7a^2-14a^2+6b-3b$
$$=-7a^2+3b.$$

12. $2x^2+3x-10x^2-12x+4x^2$.

13. $3x-7z+6y+4y+3z$.

14. $+16ab-17.4ab-1.6ab+2ab$.

Add the following polynomials:

15. $6x-7y+3z$; $4x+3y-z$.

Solution: Add the terms of one polynomial to the corresponding similar terms of the other.

Thus $6x-7y+3z+(4x+3y-z)$
$$=6x+4x-7y+3y+3z-z$$
$$=10x-4y+2z.$$

The solution may also be arranged as follows:

$$\begin{array}{r} 6x-7y+3z \\ 4x+3y-z \\ \hline 10x-4y+2z \end{array}$$

16. $a+b+c$; $a-b-c$.

17. $2a-5b+6c$; $-4a-6b-3c$.

18. x^2+xy+y^2; x^2-xy+y^2.

19. $4m+6n-5+(2m+4n+3)+(n-6)$.

20. $3x^2+3x-7+(2x^2+4x+3)+(x-3)$.

21. $-x^2-2x+4+(-x^2-3x+8)+(x^2-x+1)$.

22. $a+3(x-y)+b$; $2b-8(x-y)+4a$; $-5(x-y)+6b$.

In each of the following exercises find the sum of the left members of the equations and the sum of the right members:

23. $$\begin{array}{r} x+7y=26 \\ 2x+3y=16 \\ \hline \end{array}$$

24. $$\begin{array}{r} x-4y=1 \\ 4x+3y=40 \\ \hline \end{array}$$

25. $$\begin{array}{r} 3x\quad\;\;=8 \\ -x-4=2 \\ \hline \end{array}$$

26. $$\begin{array}{r} .8x+.2y=10.2 \\ 4x-3.5y=11.4 \\ \hline \end{array}$$

27. $$\begin{array}{r} \dfrac{2x}{3}+4y=\dfrac{26}{3} \\ 3x-\dfrac{7y}{2}=-4 \\ \hline \end{array}$$

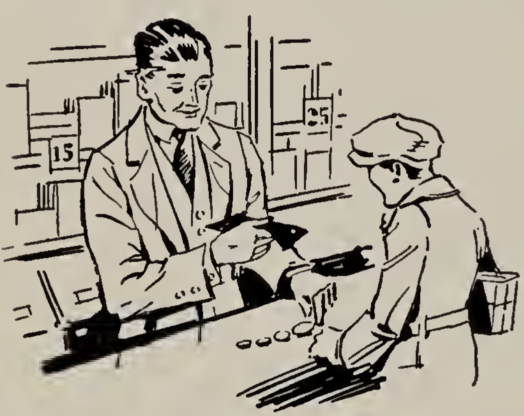
Subtraction

86. A clerk's way of subtracting. John was sent by his mother to make a purchase, for which she gave him a two-dollar bill. The price of the article was 38 cents. To make sure that he would have the correct change, he wrote on a piece of paper the following:

$$\begin{array}{r} \$2.00 \\ .38 \\ \hline \$1.62 \end{array}$$

John's way of determining the correct change was to subtract \$.38 from \$2.00.

The clerk placed the two-dollar bill in the cash drawer and then laid on the counter the following



amounts: 2 cents, saying, "38, 40," a dime, saying, "50," a half-dollar, saying, "one dollar," a dollar bill, saying, "two dollars." John counted the change and found it to be correct, \$1.62. The clerk's method of making change apparently not only gave the same amount as John's method, but was very simple. John first subtracted \$.38 from \$2 and later counted the change to see if he had the correct amount. The clerk just added enough money to \$.38 to make \$2. His way of subtracting was to find how much he had to add to the \$.38 to get a sum equal to \$2. He really avoided subtraction by changing it into addition.

87. Subtracting by means of the number scale. Since thermometer readings are signed numbers we

may subtract signed numbers by finding the *difference between two thermometer readings, i.e.*, by determining the *change* in the position of the top of the mercury column of a thermometer. If the first reading (Fig. 104) is $+16°$ and the second $+2°$ (Fig. 105) we find the difference by counting from the second reading $+2°$ to the first reading $+16°$. Since in passing from $+2°$ to $+16°$ we are counting *upward*, the difference is $+14°$. Similarly, if the first reading is $-3°$ and the second $+5°$, we find the difference by counting *downward*. Hence, the difference is $-8°$.

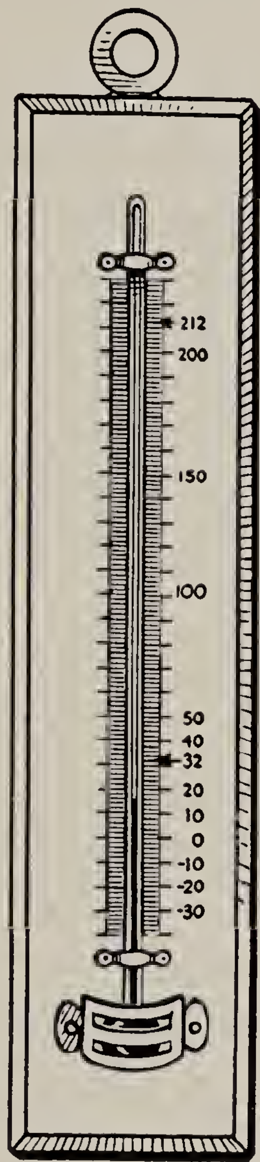


Fig. 104

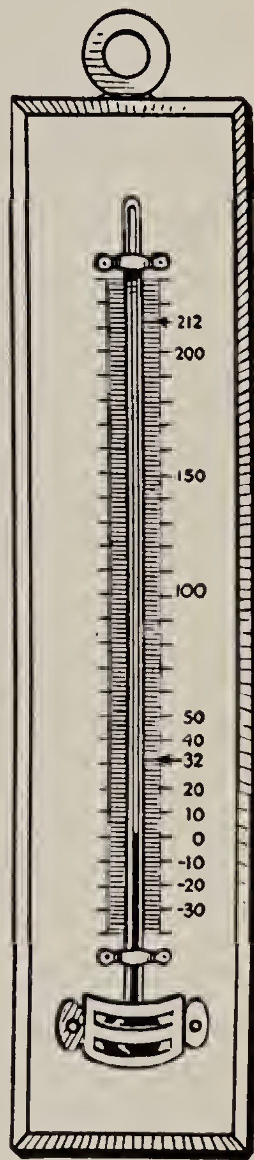


Fig. 105

It is customary to arrange the written work of subtracting in either of the following two forms:

$$+16° - (+2°) = +14°, \quad \text{or} \quad \begin{array}{r} +16° \\ +\ \ 2° \\ \hline +14° \end{array}$$

Similarly we write:

$$-3° - (+5°) = -8°, \quad \text{or} \quad \begin{array}{r} -3° \\ +5° \\ \hline -8°. \end{array}$$

The number to be *diminished*, as the $+16°$ or $-3°$ above, is the *minuend*. The number to be *subtracted*, as $+2°$ or $+5°$, is the *subtrahend*. The result, as $+14°$ or $-8°$, is the *difference*.

The following examples illustrate further the process of subtracting numbers:

1. A man who died in 1916 was born in 1876. His age is determined by *counting* from the time of birth to the time of death, which gives 40.

Thus $1916-1876=40$.

2. It is said that Augustus was born 63 B.C. (-63) and that he died 14 A.D. $(+14)$. To find his age we may lay off -63 (Fig. 106) on a number scale to the left of the zero point and $+14$ to the right. Counting from the point -63 to $+14$, we have the difference, 77.

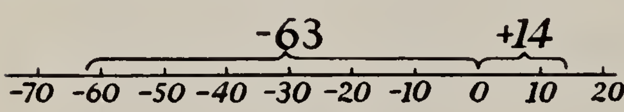


Fig. 106

This may be written $+14-(-63)=+77$.

Thus, *the difference between two numbers may be found on the number scale by counting from the subtrahend to the minuend.*

Note that here, as in making change, subtracting is really replaced by counting or adding.

Exercises

Solve the following exercises with the number scale:

1. On two successive days the average temperatures were $+8°$ and $-2°$. Find the difference.

2. The famous ancient mathematician Archimedes was born 287 B.C. and died 212 B.C. Find his age at the time of his death.

3. The longitude of a prominent building in Paris is 2°20′ East, and that of the City Hall, New York, is 74° West. Find the difference in longitude.

4. A ship sails from a position in latitude 8°30′ North to latitude 15°20′ South. Through how many degrees of latitude has it sailed?

88. A simple rule for subtracting numbers. We have seen that subtraction is the process of finding the number which added to the subtrahend gives the minuend, and that we may subtract by counting along the number scale from the subtrahend to the minuend. We are now able to find a simpler method, which will be shown from a study of the following four examples:

1. From $+2$ subtract $+5$.

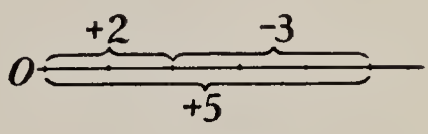


FIG. 107

Solution: On the number scale mark $+5$ and $+2$ (Fig. 107). From $+5$ count to $+2$, *i.e.*, count 3 units to the *left*. Hence, the difference is -3.

$$\text{Thus, } +2-(+5)=-3$$

$$\text{Comparing this with } +2+(-5)=-3,$$

$$\text{We find that } +2-(+5)=+2+(-5) \quad \text{. . (1)}$$

2. From -2 subtract -5.

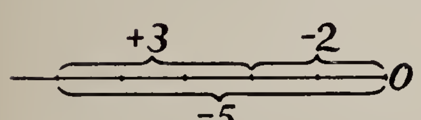


FIG. 108

Solution: As before mark off -2 and -5 (Fig. 108). Begin at point -5 and count to -2, *i.e.*, 3 units to the *right*.

$$\text{Hence } (-2)-(-5)=+3$$

$$\text{Comparing this with } (-2)+(+5)=+3,$$

$$\text{We find that } (-2)-(-5)=(-2)+(+5) \quad \text{. . (2)}$$

3. From $+2$ subtract -5.

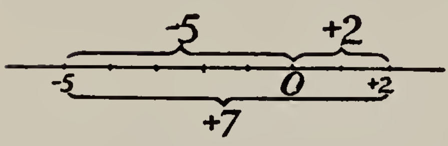


FIG. 109

Solution: Beginning at the point -5 (Fig. 109), count 7 units to the *right* to the point $+2$.

$$\begin{aligned} \text{Hence } +2-(-5) &= +7. \\ \text{But } +2+(+5) &= +7 \\ \text{Therefore } +2-(-5) &= +2+(+5) \quad \ldots\ldots\ldots \quad (3) \end{aligned}$$

4. From -2 subtract $+5$.

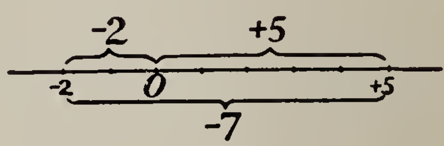


FIG. 110

Solution: Beginning at $+5$ (Fig. 110), count 7 units to the *left* to -2.

This shows that the difference is -7.

$$\begin{aligned} \text{Hence } -2-(+5) &= -7 \\ \text{Since } -2+(-5) &= -7 \\ \text{It follows that } -2-(+5) &= -2+(-5) \quad \ldots\ldots\ldots \quad (4) \end{aligned}$$

Equations (1), (2), (3), and (4) show that in every case a subtraction problem may be replaced by an addition problem which gives the same result, by using the following rule: *To subtract a number, change the sign of the subtrahend, and add the result to the minuend. The changing of the sign* should be done *mentally.*

Explain why it is possible in algebra to subtract a number from a smaller one.

EXERCISES

In Exercises 1 to 13 subtract the lower number from the upper by using the rule above, doing most of the work orally.

1. $\begin{array}{r} +6 \\ \underline{-3} \end{array}$

Solution: Change -3 mentally to $+3$, and then add $+3$ to $+6$. The result is $+9$.

2. $\begin{array}{r} +28 \\ -10 \\ \hline \end{array}$ 5. $\begin{array}{r} -11 \\ +3 \\ \hline \end{array}$ 8. $\begin{array}{r} +36 \\ +\ 9 \\ \hline \end{array}$ 11. $\begin{array}{r} -8 \\ -8 \\ \hline \end{array}$

3. $\begin{array}{r} -16 \\ -18 \\ \hline \end{array}$ 6. $\begin{array}{r} -13 \\ -\ 5 \\ \hline \end{array}$ 9. $\begin{array}{r} +58 \\ -13 \\ \hline \end{array}$ 12. $\begin{array}{r} -9 \\ +9 \\ \hline \end{array}$

4. $\begin{array}{r} +\ 3 \\ 15 \\ \hline \end{array}$ 7. $\begin{array}{r} -18 \\ +\ 4 \\ \hline \end{array}$ 10. $\begin{array}{r} +5 \\ +5 \\ \hline \end{array}$ 13. $\begin{array}{r} +24 \\ -17 \\ \hline \end{array}$

In Exercises 14 to 20 combine similar terms, doing all you can orally.

14. $+8-(3)+(-6)-(-4)+(+6)$.

Oral solution: $+8-(3) \quad =5$

$$\begin{aligned} 5+(-6)&=-1 \\ -1-(-4)&=3 \\ 3+6 \quad &=9. \end{aligned}$$

15. $3+(-6)+(+5)-(+8)-(-10)$.

16. $15a-(-6a)+(-12a)+(+8a)$.

17. $(-7m)+(+15m)-(14m)-(-6m)$.

18. $-32xy+45xy-13xy+33xy$.

19. $12a^2-8a^2+9a-6a-13$.

20. $18x^2y+16xyz-20x^2y+2xyz$.

In Exercises 21 to 24, subtract the terms of the lower polynomial from the similar terms of the upper:

21. $\begin{array}{r} 17a^2-6a+3 \\ 8a^2-4a-6 \\ \hline \end{array}$ 23. $\begin{array}{r} x^2+12x-5 \\ -2x^2+\ 7x-8 \\ \hline \end{array}$

22. $\begin{array}{r} 2ab-5bc-3ac \\ -4ab+6bc-5ac \\ \hline \end{array}$ 24. $\begin{array}{r} x^2+2xy-y^2 \\ x^2-2xy+y^2 \\ \hline \end{array}$

Subtract the lower equation from the upper:

25. $\begin{array}{r} 8x+5y=44 \\ 2x-\ y=\ 2 \\ \hline \end{array}$ 26. $\begin{array}{r} 7x+3y=-36 \\ -x+3y=\quad 7 \\ \hline \end{array}$

27. $\begin{aligned} 2x-3y&=4 \\ 2x+5y&=30 \end{aligned}$

28. $\begin{aligned} 5x+3y&=26 \\ -5x+3y&=-14 \end{aligned}$

29. $\begin{aligned} 9a-2b&=42 \\ 6a-b&=31 \end{aligned}$

30. $\begin{aligned} 8m-21n&=30 \\ 6m+35n&=17 \end{aligned}$

Multiplication

89. Multiplication of signed numbers by arithmetical numbers. If a boy deposits 4 dollars each month for 3 months he changes his bank account by 3 $(+4)$ dollars, which is an *increase* of 12 dollars.

Thus, $\mathbf{3(+4)=+12}$.

If a boy withdraws 4 dollars a month, the account is changed by 3 (-4) dollars, or a *decrease* of 12 dollars.

Thus, $\mathbf{3(-4)=-12}$.

If the mercury rises 2° each hour, in 4 hours it changes $4(+2)°$, which is a *rise* of 8°.

Thus, $\mathbf{4(+2)=+8}$.

If the mercury falls 2° each hour, in 4 hours it changes $4(-2)°$, which is a *fall* of 8°.

Thus, $\mathbf{4(-2)=-8}$.

The examples above show how to determine the sign when we multiply a positive or negative number by an *arithmetical* number. We must next learn how to multiply a signed number by a *negative* number. This will be shown below.

90. How to multiply signed numbers graphically. The four examples on the next page illustrate the process of multiplying two signed numbers by means of the number scale.

1. Multiply $+2$ by $+3$.

This is interpreted to mean that $+2$ is to be laid off three times in its own direction (Fig. 111). The result is $+6$.

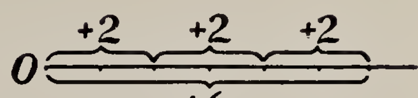


FIG. 111

Hence $(+3)(+2)=+6$ . . . . (1)

2. Multiply -2 by $+3$.

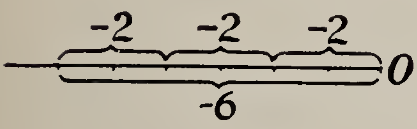


FIG. 112

This means that -2 is to be laid off 3 times in its own direction (Fig. 112). The result is -6.

Hence $(+3)(-2)=-6$ . . . . (2)

3. Multiply $+2$ by -3.

Whatever may be the meaning to be assigned to $(-3)(+2)$, it must be in agreement with the laws of algebra. In particular, it must not violate the law of order in multiplication. Accordingly we must have

$$(-3)(+2)=(+2)(-3)$$

From Example 2 it follows that the product $(+2)(-3)$ is equal to -6. Hence the product $(-3)(+2)$ must also be -6.

The same result will be obtained graphically if we interpret $(-3)(+2)$ to mean that $+2$ is to be laid off 3 times in the direction *opposite* to that of the sign of $+2$.

Thus $(-3)(+2)=-6$ . . . . (3)

4. Multiply -2 by -3.

As in Example 3, the product $(-3)(-2)$ should mean that -2 is to be laid off 3 times in the direction *opposite* to that of the sign of -2.

Hence $(-3)(-2)=+6$ . . . . (4)

91. A law for finding the sign of a product. By examining equations (1), (2), (3), and (4) in §90 we obtain the following law for multiplying signed numbers:

a. The product of two numbers having like signs is positive.

b. The product of two numbers having unlike signs is negative.

Exercises

In performing the multiplications in the following exercises, determine *first the sign*, using the laws stated above, and *then the arithmetical product*. Do most of the work orally.

1. $(-17)\ (-2)$.

Solution: The sign is $+$, the arithmetical product is 34. Hence $(-17)\ (-2)=+34$.

2. $8(-2)$.
3. $(-3)\ (6)$.
4. $(-4)\ (-7)$.
5. $(+3)\ (+6)$.
6. $(-8)\ (+4)$.
7. $-8(-7)$.
8. $(+6)\ (4)$.
9. $2(-19)$.
10. $-5(+21)$.
11. $(-3)\ (-15)$.
12. $6(+5)$.
13. $+8(-3)$.
14. $(-15)\ (13)$.
15. $(-6)\ (-4)$.
16. $(+12)\ (+8)$.
17. $(\frac{2}{9})\ (-\frac{3}{10})$.

Solution: $(\frac{2}{9})\ (-\frac{3}{10})=-\dfrac{2\times \cancel{3}}{\underset{3}{\cancel{9}}\times\underset{5}{\cancel{10}}}=-\frac{1}{15}$.

18. $(-\frac{2}{15})\ (+\frac{5}{8})$.
19. $(-\frac{7}{8})\ (-\frac{4}{15})$.
20. $(+6\frac{3}{4})\ (-\frac{4}{9})$.
21. $(-4.8)\ (2.8)$.
22. $(6\frac{1}{3})\ (-\frac{9}{38})$.
23. $(-\frac{7}{12})\ (-\frac{15}{28})$.
24. $(-2)\ (3)\ (-4)$.

Solution: $(-2)\ (3)\ (-4)=+2\times 3\times 4=24$.

25. $-6(-2)\ (-5)$.
26. $(-4)\ (+8)\ (-3)$.
27. $(+10)\ (-6)\ (3)$.
28. $(8)\ (-4)\ (-2)$.

29. $(-3x^2)\ (-2y^3)\ (+xy)$.

Solution: The sign is $+$.
The arithmetical product is $3\times2\times1$.
The literal product is x^2y^3xy, or x^3y^4.
Hence $(-3x^2)\ (-2y^3)\ (xy)=6x^3\ y^4$.

30. $a(-b)\ (-c)$.

31. $-a(2b)\ (-3c)$.

32. $mn(-2m^2)\ (3mn)$.

33. $x(-y^2)\ (-2x^2y)$.

34. $(4a)\ (-3n)\ (-bc)$.

35. $(-5x)\ (-3x)\ (-7b)\ (-3b)$.

36. $(-2)\ (-\frac{3}{2}a)\ (a^2b)\ (-\frac{5}{3})$.

37. $-(2x)\ (-2y)\ (3xy)$.

38. $\frac{3}{2}(a^2b)\left(-\frac{ab}{3}\right)(-4)$.

39. $-2(x-3)$.

Solution: $-2(x-3)=-2x+6$.

40. $-a(a+5)$.

41. $-3x(x-y)$.

42. $(5y^2+3y+2)\ (-y)$.

43. $(3x^2-7x-4)\ (-6x)$.

44. $(x+3)\ (x-5)$.

Solution: $(x+3)\ (x-5)=x^2-5x+3x-15$
$=x^2-2x-15$.

45. $(a-8)\ (a-2)$.

46. $(2x+4)\ (3x-5)$.

47. $(3x-7y-7z)\ (2x-3y+4z)$.

48. $(2x^2+5x-1)\ (-x^2+x-4)$.

Find the value of each of the following:

49. $3x^3-2x^2-x+3$ when $x=-2$

Solution: $3x^3-2x^2-x+3=3(-2)^3-2(-2)^2-(-2)+3$
$=3(-8)-2(4)+2+3$
$=-24-8+5$
$=-27$.

50. x^3+5x^2+2x-8 when $x=-3$.

51. $2x^3-6x^2+3x-4$ when $x=-\frac{1}{3}$.

52. $-3x^3+2x^2-8x-7$ when $x=-1$.

53. $-x^3-8x^2+2x+3$ when $x=-\frac{1}{2}$.

Division

92. The law of signs in division. The process of division is the opposite of multiplication, *i.e.*, to divide 8 by 2 is to determine the number which multiplied by 2 gives 8. Thus, $\frac{6}{2}=3$, because $3\times 2=6$.

Applying this meaning of division to signed numbers, we have the following:

$$\frac{+6}{+2}=+3, \text{ because } (+3)\ (+2)=+6$$

$$\frac{-6}{-2}=+3, \text{ because } (+3)\ (-2)=-6$$

$$\frac{+6}{-2}=-3, \text{ because } (-3)\ (-2)=+6$$

$$\frac{-6}{+2}=-3, \text{ because } (-3)\ (+2)=-6.$$

These four examples illustrate the following law of signs in division:

a. The quotient of two numbers having like signs is positive.

b. The quotient of two numbers having unlike signs is negative.

Exercises

In performing the indicated divisions below, determine (1) the sign, (2) the arithmetical quotient, and (3) the literal quotient:

1. $\dfrac{144m^4n^6}{-4m^4n^2}$

Solution: $\dfrac{\overset{36}{\cancel{144}}\cancel{m^4}\overset{n^4}{\cancel{n^6}}}{-\cancel{4}\cancel{m^4}\cancel{n^2}}=-36n^4.$

2. $\dfrac{21}{-3}$

3. $\dfrac{+10}{-5}$

4. $\dfrac{625}{-25}$

5. $\dfrac{-34.3}{-0.7}$

6. $(-90) \div (+45)$.

7. $8(45) \div (-15)$.

8. $196 \div (-14)$.

9. $-275 \div (-25)$.

10. $(-24) \div (8)$.

11. $(90) \div (-15)$.

12. $(-144) \div (-72)$.

13. $(-a) \div (-a)$.

14. $(1.21) \div (-11)$.

15. $(-2.25) \div (4.5)$.

16. $\dfrac{20y^2}{-5y}$.

17. $\dfrac{-ar^2}{+a^2r}$.

18. $-\dfrac{-a^2y}{+ay^3}$

19. $\dfrac{3^2 \cdot 5^2 \cdot 7^2}{-5^2 \cdot 7^3}$.

20. $-\dfrac{-13p^3q}{-26p}$.

21. $\dfrac{-8m^4n^2}{-2mn^2}$

22. $\dfrac{96a^4b^4}{-6ab}$.

23. $\dfrac{-9x^6}{-27x^4}$.

24. $\dfrac{(+2)^3y}{(+2)^2y^2}$

25. $(-\frac{4}{3}a^2) \div (-\frac{20}{21}a)$.

Solution: $(-\frac{4}{3}a^2) \div (-\frac{20}{21}a) = (\frac{4}{3}a^2) \times (\frac{21}{20}a)$

$$= \frac{\overset{a}{\cancel{4}\cancel{a^2}} \times \overset{7}{\cancel{21}}}{\cancel{3} \times \underset{5}{\cancel{20}\cancel{a}}} = \frac{7a}{5}.$$

26. $(-\frac{5}{6}) \div (-\frac{2}{3})$.

27. $(-\frac{3}{4}x^2y) \div (-\frac{1}{2}x^3y^2)$

28. $(-7\frac{2}{3}yz^2) \div (11\frac{1}{2}z)$.

29. $(-3\frac{2}{5}ab) \div (\frac{34}{15}a^2b)$.

Solve the following equations and check each:

30. $6x+12=3x-3$.

Solution: $6x+12=3x-3$.
Subtracting $3x$, $3x+12=-3$
Subtracting 12, $3x=-15$
Dividing by 3, $x=-5$.

Check:	LEFT SIDE		RIGHT SIDE
	$6(-5)+12$		$3(-5)-3$
	$-30+12$		$-15-3$
	-18	$=$	$-18.$

31. $7x-16=-100.$

32. $-4x+8=20.$

33. $5-2x=19.$

34. $18-3x=6x+81.$

35. $5x-3=8x+15.$

36. $10x+20=9x+16.$

37. A square is to be changed into a rectangle having the same area as the square by making one side 12 feet longer and the other 4 feet shorter. What are the dimensions of the rectangle?

Change the following fractions to the simplest form:

38. $\dfrac{18m^2n-27mn^2}{9mn}$

Solution: $\dfrac{18m^2n-27mn^2}{9mn}=\dfrac{\cancel{9mn}(2m-3n)}{\cancel{9mn}}=2m-3n.$

39. $\dfrac{49ab^3-35a^2b^2}{7ab^2}$

40. $\dfrac{-a^4b-ab^4-5a^2}{a^2}.$

41. $\dfrac{-3a^2y+27ay^3}{-3ay}$

42. $\dfrac{-4m^3n+12m^2n^2}{-2m^2n}.$

93. What every pupil should be able to do. Having studied Chapter V, you should be able to do the following:

1. To add, subtract, multiply, and divide signed numbers with accuracy and a fair degree of speed.
2. To add numbers graphically and by rule.
3. To subtract numbers by changing the sign of the subtrahend and adding the result to the minuend.
4. To use the laws of signs in multiplying and dividing.

5. To solve any equation of the form $5x-3=2x-7$, where any positive or negative number may be taken as coefficient.

6. To add and subtract polynomials.

7. To multiply polynomials.

94. Typical exercises. The exercises below are typical of the work of the chapter. Every pupil should be able to work them.

Perform the following operations and explain each:

1. $+6+(-8)$.
$-5+(-2)$.

2. $-3-(-7)$.
$+5-(+3)$.

3. $(+3)\ (+2)$.
$(+5)\ (-3)$.
$(-3)\ (-4)$.
$(-6)\ (+5)$.

4. State the laws of signs and illustrate each with one example.

Perform the operations indicated below:

5. $15a-(-3a)+(-5a)-(2a)$.

6. $12x^2y+(-3z)-(6z)-(8x^2y)$.

7. $(16a^2-3a+5)+(10a^2-2a-3)$.

8. $(2xz+3yz-8xy)-(6xz-4yz+6xy)$.

9. $(-8)\ (-5)\ (-2)$.

10. $(-2)\ (-3)\ (+4)$.

11. $\left(-\frac{2}{15x}\right)\left(+\frac{5}{8xy}\right)$.

12. $(4a)\left(-\frac{3}{2m}\right)\left(-\frac{5}{an}\right)$.

13. $(-625)\div(-25)$.

14. $(2x^2-17x)4x$.

15. $(x^2-3xy+y^2)\ (2xy)$.

16. $(2a-b)\ (a+2b)$.

17. $(8a-3y)\ (a-y)$.

18. $(2a^2+7a-9)\ (5a-1)$.

19. $\frac{-8\ m^4np^2}{-26m^2n^3p}$.

20. $\left(\frac{4x^4}{5}\right)\div(2x^2)$.

21. $\frac{7a}{15}\div\frac{14a^2}{20}$.

In the following exercises first add the upper to the lower, then subtract the lower from the upper equation:

22.
$$\begin{array}{r} 6x-3y=14 \\ -3x-2y=-\ 8 \\ \hline \end{array}$$

23.
$$\begin{array}{r} 3x+\ y=1 \\ -4x+3y=7 \\ \hline \end{array}$$

Solve the following equations:

24. $2x+16=4.$

25. $19x-4=12x-18.$

CHAPTER VI

SOLVING SIMPLE EQUATIONS AND PROBLEMS

What You Already Know about Equations

95. The importance of knowing how to solve equations. We know from our former work in mathematics that the equation is a useful tool for solving problems. Let us recall several examples to illustrate this statement.

1. A triangle is to be constructed in which the first angle is to be three times the second, and the third six times the second.

The principle that the sum of the angles of a triangle is 180° enables us to express the facts stated in the problem in the form of the equation $3x+x+6x=180$. By solving the equation we can find the solution of the problem.

2. If our boat runs 30 miles in $3\frac{1}{4}$ hours, how far shall we travel in 8 hours?

We know that the distance is proportional to the time. The relation between the distance and time can be expressed precisely by the equation

$$\frac{30}{3\frac{1}{2}}=\frac{x}{8}.$$

The problem is solved by solving the equation.

3. John solved twice as many problems as James. Mary solved three times as many as James. Together they solved 48 problems. How many problems did each solve?

In this problem the equation may be used to express in two different ways the total number of problems solved, *i.e.*, $x+2x+3x$ and 48. Equating these two number expressions we have $x+2x+3x=48$, which when solved determines the value of x.

4. In problems of perimeters, areas, and volumes the formulas lead to equations from which we obtain the solutions of the problems.

Many other examples could be given to show how valuable the equation is in finding the solution of problems.

96. How equations are solved. Let us review briefly some simple equations we have learned to solve.

We found the first equations when we studied perimeters in problems of this type: Find the side of an equilateral hexagon whose perimeter is 120. The equation is $6x=120$. This equation is very easily solved. In fact, we can tell by inspection that x must be 20. However, let us, in finding the result, call attention to the algebraic laws that are used. The solution in full should then be as follows:

$$6x=120$$

Dividing both members (sides) of the equation by 6, we have

$$\frac{\cancel{6}x}{\cancel{6}}=\frac{120}{6}$$

Changing the fractions to simplest forms,

$$x=20$$

Using this method of solving, determine the unknown numbers in the following exercises:

EXERCISES

1. $1.23y = 532$.
2. $7.5m = 28.2$.
3. $3.14d = 4.71$.
4. $.25p = 938$.
5. $3.5x = 70$.
6. $1.06a = 530$.
7. $3.14d = 785$.
8. $.75a = 18$.
9. $.57x = 24.2$.
10. $.231a = 462$.

11. Draw two lines (Fig. 113) making one of the adjacent angles four times as large as the other.

Solution: Show that $x + 4x = 180$

Combining similar terms we have

$$5x = 180$$

Dividing both members by 5,

$$x = 36$$

$$\text{and } 4x = 144.$$

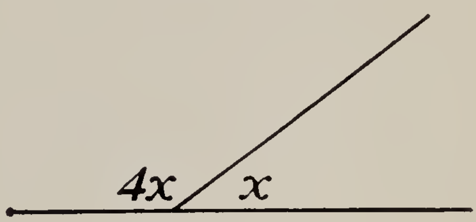


FIG. 113

12. $x + 3x = 24$.
13. $p + .45p = 43.5$.
14. $5x + 2x = 28$.
15. $8y - 2y = 12$.
16. $11x - 9x = 16$.
17. $7b + 8b = 45$.

97. Directions for solving simple equations. We have learned that in solving an equation like

$$50 - 6n = n - 20$$

we must first *bring all terms containing the unknown number* n *to one side of the equation.* This is done by adding, or subtracting, the same number to both sides of the equation. In the equation above we may add $6n$ to both sides, which gives the equation

$$50 = 7n - 20.$$

The next step is to *bring all terms not containing the unknown number to the other side.* Adding 20 to both sides, we have

$$70=7n.$$

Finally, *divide both sides by the coefficient of* n. The result is

$$10=n, \text{ or } n=10.$$

Exercises

Solve the following equations following the directions given in §97:

1. $16-9n=9n-2$.

Solution: $16-9n=9n-2$

$16=18n-2$, by adding 9 to both sides

$18=18n$, by adding 2 to both sides

$1=n$, by dividing both sides by 18

Hence $n=1$.

Check:

Left Member		Right Member
$16-9$		$9-2$, by substituting 1 for n in the original equation.
7	$=$	7, by combining terms.

2. $8x-9=-2x+11$.

3. $-5a=-2a-39$.

4. $6x+40-11x=0$.

5. $-12b+18=6b$.

6. $4m-7=53-6m$.

7. $17-8s=2s-47$.

8. $11x-9=5x+117$.

9. $42-3x=48-9x$.

10. $5x-25=3x-5$.

11. $2x+21=58-2x$.

12. $2m-17+m-34=-54$.

13. $2s+3=-s+1+2s-5$.

14. $-2y+1=-4y+3$.

15. $2-64r=144+7r$.

16. $4a-15-a=35-2a$.

17. $9x+10=88+2x-8$.

Translating Verbal Statements into Symbols

98. How to derive equations from verbal problems. Since many problems are solved by means of equations, you must learn (1) how to *derive* the equation, or equations, and (2) how to *solve* the equations. The second part no longer offers a serious difficulty because you have definite directions for solving equations (§97). For the first, no general directions can be given, but the following simple suggestions will be helpful as they apply to all verbal problems. Additional special directions, to be given later (§99 to 106), will apply to certain types of problems designed to give you practice in deriving the equation from a problem.

1. *Read the problem carefully.* The purpose of this is to get general information as to the content of the problem.

2. *Read the problem again to determine what it asks for.* The number or numbers to be found are the *unknown* numbers.

3. *Denote one, or several, of the unknown numbers by letters.* Make a definite statement as to what these letters stand for. Thus, if the *rate* of motion is to be found, write: "Let r be the number of miles per hour," or "the number of yards per minute," or "the number of feet per second." Do not say briefly "Let r be the rate" because this statement does not indicate the *unit.* Similarly, if the *price* is called for, write: "Let n be the number of cents" or "Let n be the number of dollars," not "Let n be the price."

4. *Read the problem again, one sentence at a time, and express the various facts it contains in terms of the*

unknown literal number, or numbers. For example, if the first sentence reads "A sum of \$5330 is to be divided into two parts," the facts are stated as follows:

Let x be the number of dollars in the first part.

Let y be the number of dollars in the second part.

Then $x+y=5330$.

Note that this equation is obtained by equating two number expressions $x+y$ and 5330 which denote the same sum.

If only one unknown number is to be used, we should write:

Let x be the number of dollars in the first part.

Then $5330-x$ is the number of dollars in the second part.

5. *When the data of the problem have been translated into symbols, state the equation.* The following suggestions will usually help you to state the equation.

a. Translate the verbal problem into symbols. For example, the statement: "A number diminished by 20 is equal to 50 decreased by 6 times the number," when translated into symbols, gives the equation

$$n-20=50-6n.$$

The equation is simply an abbreviated form of the verbal statement.

b. State a formula, or a principle, relating the facts given in the problem. To illustrate, let x, $2x$, and $3x$ denote the number of degrees in the angles of a triangle. The principle that the sum of the angles of any triangle is 180° gives the equation

$$x+2x+3x=180.$$

c. Equate two number expressions denoting the same fact. Thus, if two trains traveling at different rates are equally distant from the station from which they started, and if they have traveled, respectively, $5x$ and $8(x-6)$ miles, the equation is

$$8(x-6)=5x.$$

It states that both trains are the same number of miles from the station.

Exercises

Translate the following statements into algebraic symbols:

1. A number decreased by 85 is equal to 99.

2. Four times a number decreased by 2 is equal to 3 times the number increased by 6.

3. Five times a number diminished by the number is 10 greater than 2 times the number.

4. Four times a number decreased by 3 exceeds 2 times the number by 15.

5. One-third of a number increased by 8 is equal to 38 diminished by 3 times the number.

6. One-fourth of a number added to one-fifth of the number is one less than one-half of the number.

7. The base of a triangle exceeds the altitude by 4 inches.

In Exercises 8 to 11 find the equations from a general principle, or a formula. Do not solve the equations.

8. The hypotenuse of a right triangle is 10 feet and one of the sides is 3 feet longer than the other.

9. A tree was broken over so that the top touched the ground 40 feet from the foot of the stump. The stump was 12 feet high. Find the height of the tree.

10. One angle of a triangle is 5° greater than 5 times another, and the third angle is 20°.

11. A bird flies a distance of 90 miles in $2\frac{1}{2}$ hours. Find the rate.

In Exercises 12 to 15 state the equation by equating two expressions denoting the same number:

12. A man rows downstream $\frac{x}{6}$ hours and returns in $\frac{x}{3}$ hours, using 9 hours for the complete trip.

13. A merchant lost $350 in an investment. His loss was 9 per cent of the amount invested.

14. The sides of a rectangular field are x and $2x$ rods. By making the rectangle 20 rods longer and 24 rods wider the area is doubled.

15. A freight train traveling x miles an hour is overtaken $1\frac{3}{5}$ hours after it started by an express train which left the station one hour later than the freight train and which travels 40 miles an hour.

Practice Problems in Deriving and Solving Equations

99. Perimeter problems. The *perimeter* of a polygon is the sum of the lengths of the sides. In each of the following problems express all unknown numbers in terms of one letter, derive the equation, and solve.

Exercises

1. To inclose a playground 200 rods of wire fencing are available. If the length is to be $1\frac{1}{2}$ times as great as the width, find the dimensions.

Solution: Let w be the number of rods in the width (Fig. 114).

Then $\frac{3w}{2}$ is the number of rods in the length.

and $\frac{3w}{2}+w$ is the half-perimeter.

Hence $\frac{3w}{2}+w=100$, by equating two numbers expressing the half-perimeter.

$2\left(\frac{3w}{2}\right)+2w=2\times100$, by multiplying each term by 2.

$\therefore 5w=200$

$w=40$

and $\frac{3w}{2}=60$.

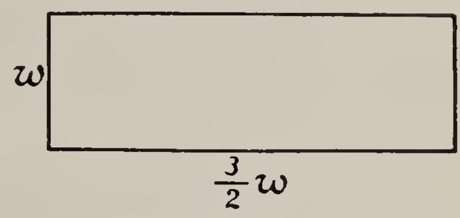


FIG. 114

2. A triangular piece of ground is to be laid off with one side twice as long as the second, and the third side 3 times as long as the second. What must be the lengths of the sides, if the perimeter is to be 72 rods?

3. Find the lengths of the sides of an isosceles triangle (having 2 equal sides) whose base is to be 60 feet, and whose perimeter is to be 240 feet.

4. The length of a rectangular field is 3 times the width. The perimeter is 264 feet. Find the dimensions.

5. The width of a rectangle is to the length as 3 is to 5. The difference between the length and width is 8 feet. Find the dimensions.

Suggestion: Let $3x$ be the number of feet in one side.
Then $5x$ is the number of feet in the other side.

6. The length of a rectangle exceeds the width by 14 feet. If the perimeter is 240 feet, determine the dimensions.

7. The perimeter of a rectangle is 184 feet and the width is 8 feet less than the length. Find the dimensions.

8. A rectangle is 4 feet longer than twice the width. Find the dimensions if the perimeter is $13\frac{1}{6}$ feet.

9. A rectangle is 8 feet longer than twice the width. If the perimeter is 232 feet what are the dimensions?

10. A rug is 2 feet longer than it is wide. The sum of the length and width is 14 feet. Find the dimensions.

100. Problems stating number relations. The following exercises should be worked with one unknown number:

EXERCISES

1. A man left \$18,500 to his wife and son. The mother was to receive three times as much money as the son. How should the money be divided?

2. A man owns a lot and has saved \$6000 with which to build a home. He can borrow from a bank an amount of money equal to one-third of the cost of the house. What is the largest amount of money he can spend on the house?

3. Three men plan to buy a business costing \$8600. One has \$1700. The remainder is to be furnished by the others so that one pays twice as much as the other. How much money does each furnish?

4. Find two numbers whose sum is 23 and whose difference is 5.

5. Find two parts of 90 so that one part exceeds one-half the other by 20.

6. Find two parts of 240 so that twice the larger part exceeds 5 times the smaller by 11.

7. Find two numbers whose difference is 36, if one is 3 times as large as the other.

8. A man has \$320 to spend for repairs on his house. The materials needed cost \$50. How much per day will a carpenter be able to earn, if it takes him and his helper 15 days to do the work and if he is to earn 4 times as much as the helper?

101. Age problems. The following exercises should be worked with one unknown number:

EXERCISES

1. If a boy's age is 13 years, how old will he be in 3 years? In five years? In x years? How old was he 4 years ago? y years ago?

2. If a man's age is x years, how old was he 8 years ago? How old will he be in 5 years?

3. A is twice as old as B, but 7 years ago A was 3 times as old as B. Find the present age of each.

4. A father is twice as old as his son. Ten years ago he was 3 times as old as the son. What is the present age of each?

5. Eight years ago A was 6 times as old as B. Seven years from now A will be 3 times as old as B. Find the present ages.

102. Motion problems. Uniform motion depends upon distance, time, and rate. If an automobile travels 20 miles in one hour, it is said to travel at the *rate* of 20 miles per hour. Rate is sometimes called speed, or velocity, and means a distance passed over in a unit of time.

Exercises

1. Find the distance passed over by an object moving at a uniform rate and in a given time.

The numerical facts may be conveniently arranged as in the table below:

Rate	20 mi. an hr.	18 ft. a sec.	3 yd. a min.	25 mi. an hr.	r mi. an hr.
Time	4 hr.	$2\frac{1}{2}$ sec.	$18\frac{1}{2}$ min.	24 hr.	t hr.
Distance					

2. In the table below determine the rates, having given the time and distance:

Time	3 hr.	$1\frac{1}{2}$ hr.	4 hr.	2 sec.	t hr.
Distance	10 mi.	26 mi.	15 mi.	90 ft.	d mi.
Rate					

3. In the table below find the time, having given the distance and rate:

Distance	258 mi.	20 yd.	58 mi.	16 ft.	d ft.
Rate	30 mi. an hr.	3 yd. a sec.	40 mi. an hr.	6 ft. a sec.	r ft. a sec.
Time					

4. A carrier pigeon flew 70 miles in $1\frac{1}{2}$ hours. How fast did it travel?

5. The sound of a stroke of lightning was heard 8 seconds after the flash was seen. How far away was the stroke if sound travels at a rate of 1080 feet a second?

6. Two stations are 32 miles apart. Two trains leaving the stations at the same time travel toward each other at the rate of 30 and 50 miles an hour respectively. How soon after starting will they meet?

Solution: Instead of writing out in full the given facts of the problem we may arrange them briefly in the form of a table as shown below. The distances are derived from the given rates and the unknown time.

	First Train	*Second Train*
Time in hours	x	x
Rates in miles per hour	30	50
Distances in miles..................	$30x$	$50x$

The equation is obtained by stating the fact that the stations are 32 miles apart (Fig. 115).

$$\begin{aligned} \text{Thus } 30x+50x&=32 \\ 80x&=32 \\ x&=.4 \end{aligned}$$

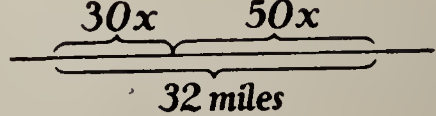


Fig. 115

7. Two men starting from the same place travel in opposite directions, one going twice as fast as the other. In 5 hours they are 300 miles apart. Find the rate of travel of each.

Suggestion: Arrange in the form of a table the given time, the unknown rates, and the distances derived from them.

8. Two men living 96 miles apart travel toward each other at rates of 18 miles and 20 miles an hour. If A leaves home an hour earlier than B, when will they meet?

Solution: Verify the following table:

	A	B
Time in hours	x	$x-1$
Rates in miles per hour	18	20
Distances in miles	$18x$	$20(x-1)$

State the equation and solve.

9. A train traveling at the rate of 30 miles an hour is followed by a second train traveling 35 miles an hour. If the second train leaves a station 3 hours later than the first, in how many hours will it overtake the first?

10. A freight train leaves a station and travels at a rate of 32 miles an hour. An hour later it is followed by an express train traveling 60 miles an hour. When and how far from the station will the express train pass the freight train?

11. A train leaves Buffalo for New York and travels at a rate of 30 miles per hour. Three hours later another train follows, making 50 miles an hour. When and how far from Buffalo will the second train overtake the first?

103. Mixture problems. By a "5 per cent solution" of water and salt is meant a mixture 5 per cent of which is salt. The following exercises show how to determine the amount of liquid needed to reduce a mixture to a desired solution:

Exercises

1. A druggist wishes to dilute a 25 per cent mixture of water and listerine to a 15 per cent mixture. How much water must be added to 8 ounces?

Solution: Tabulate the facts as follows:

	25% Mixture	*15% Mixture*
Number of ounces in mixtures........	8	$8+x$
Number of per cent of listerine......	25	15
Number of ounces of listerine........	$\frac{25\times8}{100}$	$\frac{15(8+x)}{100}$

Show that the equation is $\frac{15(8+x)}{100}=\frac{25\times8}{100}$.

Multiply both members of the equation by 100.

Solve the resulting equation.

2. How much water must be added to 8 gallons of milk containing 5 per cent butter fat to change it to test 4 per cent butter fat?

3. How much water must be added to a quart of a 20 per cent solution of ammonia to reduce it to a 10 per cent solution?

4. How much water must be added to 12 gallons of milk testing $5\frac{1}{4}$ per cent butter fat to change it to a mixture testing 4 per cent butter fat?

5. A druggist wishes to reduce 12 ounces of medicine containing 25 per cent alcohol to one containing 20 per cent alcohol. How much water must he add?

104. Equations containing parentheses. In solving the equations below consider carefully the operations indicated. Then decide upon the correct order in which to perform these operations.

Exercises

Solve the following equations:

1. $x-2(3-4x)=12$.

Solution: The parentheses in this equation indicate that $3-4x$ is to be multiplied by -2.

Hence $x-6+8x=12$

$$9x-6=12$$

$$\therefore x=2.$$

2. $6a-3(3a-1)-1=0$.
3. $2(x-1)=3x-6(2x+3)$.
4. $2y-2(6y-17)=3(2y-6)$.
5. $3a-2(a+5)=6a-20$.
6. $3(a-2)+15=5a-3$.
7. $8(3-2y)-2(5-y)=28$.

105. Interest problems. The interest formula is $i=\frac{prt}{100}$, where p is the sum invested, r the number of per cent, and t the number of years. The percentage formula is $p=\frac{rb}{100}$, where the number b denotes base, and p percentage. Using these formulas solve the following problems:

Exercises

1. At what rate will \$8000 yield an interest of \$910 in one year and nine months?

Solution: The table below states the facts of Exercise 1.

Principal	8000
Number of per cent. . . .	x
Time	$1\frac{3}{4}$
Interest.	$\frac{8000\times x\times 1\frac{3}{4}}{100}$

$$\text{Show that } \frac{8000\times x\times 1\frac{3}{4}}{100}=910$$

Solve the equation.

2. In how long a time will the interest on \$600 at 6 per cent amount to \$48?

3. Find the rate at which \$3645 gives \$340.20 interest in one year and four months.

4. A boy's parents plan to invest a sum of money at 6 per cent interest to yield an income large enough to cover his expenses at college. How large must this sum be if his expenses average \$100 a month for nine months?

5. What sum invested at 5 per cent will amount to \$5500 in one year and 10 months?

6. An amount of money invested at 5 per cent yields in one year an income \$300 less than twice as large a sum invested at 4 per cent. Find the amount invested at each rate.

Solution: Verify the following table:

	First Sum	*Second Sum*
Principal	x	$2x$
Number of per cent	5	4
Interest	$\frac{5x}{100}$	$\frac{8x}{100}$

Show that $\frac{5x}{100}=\frac{8x}{100}-300$.

Multiply each term by 100, reduce, and solve the resulting equation.

7. A man has a yearly income of \$54 from an investment of \$1000. If one part of the investment yields 6 per cent and the remainder 5 per cent, find the amount invested at each rate.

Suggestion: Let x be the sum invested at 6 per cent. Then $1000-x$ is the sum invested at 5 per cent.

8. A man invests part of \$10,000 at 6 per cent interest and the remainder at 5 per cent. The total yearly income is \$570. Find the amount invested at each rate.

9. A man invests two sums at 5 per cent and at 4 per cent respectively. From this investment he has a yearly income of \$500. If the total sum invested amounts to \$12,000, how much is invested at each rate?

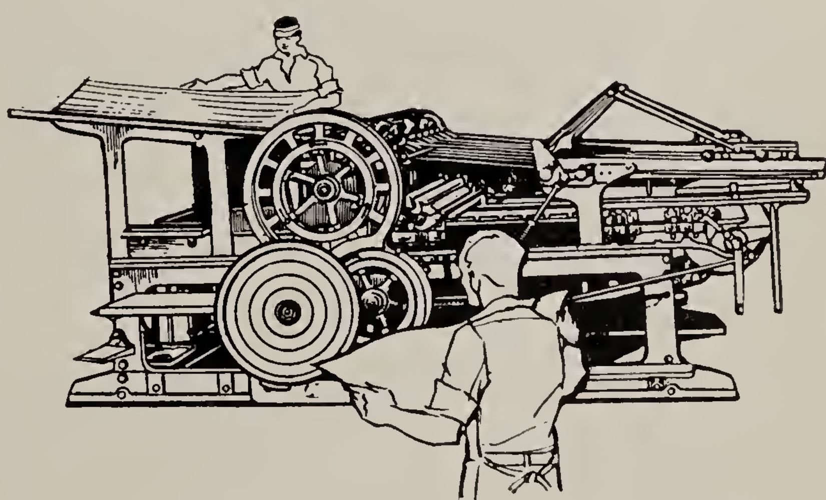
10. A sum of \$1400 is divided into two parts. The total annual income is \$70, if one part is invested at 5 per cent and the other at 6 per cent. Find the two parts.

11. A sum of \$1200 is divided into two parts. The first part, invested at 5%, yields an income \$39.10 greater than that of the second part, invested at 4.5%. How much money is invested at each rate?

106. Work problems. To determine the amount of work done within a given time, we must know the amount done in the *unit* of time, as a day, an hour, or a minute. Thus, if a machine does a complete piece of work in four hours, it does one-fourth of it in *one* hour. From this it is possible to find the amount done in *any* given number of hours. This principle may be used in the solution of the following exercises.

EXERCISES

1. A machine can print the issue of a paper in two hours, and another machine can do it in three hours. In how many hours can an edition be printed with both machines?



Solution: The facts involved in this problem are tabulated as follows:

	First Machine	*Second Machine*
Number of hours it takes to do all the work	2	3
The amount done in one hour	$\frac{1}{2}$	$\frac{1}{3}$
The amount done in x hours	$\frac{x}{2}$	$\frac{x}{3}$

Denoting the whole piece of work done by 1, show that

$$\frac{x}{2}+\frac{x}{3}=1$$

In solving this equation, multiply first every term by the least common denominator 6.

2. If a man can plow a field in 10 days, and another man can do it in 8 days, how long will it take them if they work together?

3. A carpenter can build a fence in 6 days, and his apprentice can do it in 10 days. In how many days can they do it together?

4. A can do a piece of work in 8 days and B can do it in 12 days. In how many days can they do it together?

5. One pipe can fill a tank in 11 hours. Another can fill it in 3 hours. How long will it take both to fill it?

107. How to solve fractional equations. When some or all of the terms in an equation are fractions, the solution is usually simplified by multiplying every term of the equation by the least common multiple of the denominators.

Exercises

Solve the following equations and problems:

1. $x+\frac{x}{3}=60.$

Solution: $x+\frac{x}{3}=60.$

Multiplying each term by 3, $3x+x=180$
Combining terms, $4x=180$
Dividing by 4, $x=45$

Check: Left Member		Right Member
$45+\frac{45}{3}$		60
$45+15$		60
60	=	60

2. $x-\frac{x}{50}=21$

4. $\frac{x}{2}+x=96.$

3. $a-\frac{a}{2}=27.$

5. $2x-8=\frac{2x}{3}.$

6. $\frac{x}{2}+\frac{x}{5}=35.$

Solution: $\frac{x}{2}+\frac{x}{5}=35.$

Multiplying by 10, $5x+2x=350$
Combining terms, $7x=350$
Dividing by 7, $x=50$

7. $\frac{x}{3}+\frac{x}{4}=21.$

9. $\frac{x}{4}+\frac{x}{8}=3.$

8. $\frac{x}{3}=9+\frac{x}{6}.$

10. $\frac{x}{3}-\frac{x}{10}=\frac{7}{2}.$

11. $\frac{4x}{5}-\frac{3x}{10}=50.$

Solution: $\frac{4x}{5}-\frac{3x}{10}=50$

Multiplying by 10, $8x-3x=500$
Combining terms, $5x=500$
Dividing by 5, $x=100.$

12. $\frac{3x}{4}-\frac{x}{8}=x-3.$

13. $\frac{26y}{3}+\frac{5y}{2}=67.$

14. $\dfrac{2m}{3}+\dfrac{3m}{4}=23.$

15. $\dfrac{3m}{4}+\dfrac{2m}{5}=35-\dfrac{3m}{5}.$

16. The angles $2(a+10)$ and $\dfrac{3a+68}{2}$ are supplementary. Find each angle.

17. Two opposite angles formed by two intersecting lines are denoted by $\dfrac{b}{3}+\dfrac{b}{6}$ and $18+\dfrac{b}{4}$. Find b and the angles.

18. The acute angles of a right triangle are $x+\dfrac{x}{8}$ and $\dfrac{x}{12}+17\frac{1}{2}$. Find x and each angle.

19. One angle of a triangle is 15° larger than the second. The third angle is one-sixth as large as the second. Find each angle.

20. $\dfrac{2x+3}{5}-\dfrac{x-2}{3}=\dfrac{7}{5}.$

Solution: $$\frac{\overset{3}{\cancel{15}}(2x+3)}{\cancel{5}}-\frac{\overset{5}{\cancel{15}}(x-2)}{\cancel{3}}=\frac{\overset{3}{\cancel{15}}(7)}{\cancel{5}}.$$

$$\therefore 6x+9-5x+10=21.$$

Note especially that the sign of the fourth term is +. It is a very common error to overlook the fact that both terms in the number $(x-2)$ are multiplied by -5. The product is $-5x+10$ and not $-5x-10$.

21. $\dfrac{3x-2}{7}-\dfrac{1-4x}{3}=8\frac{4}{21}.$

22. $\frac{1}{2}(5y-3)+\frac{1}{3}(5y-2)=5.$

23. $\dfrac{3x-1}{5}-\frac{1}{2}(x+6)=\frac{1}{2}.$

24. $\dfrac{a+13}{13}-\dfrac{6-3a}{65}=1.$

25. $\dfrac{2x}{31}-\dfrac{x+11}{62}=\frac{1}{31}-x+\dfrac{3x-6}{2}.$

26. $\dfrac{12+y}{2y}=\dfrac{12+2y}{3y}.$

Suggestion: Multiply both members by $6y$.

27. $\frac{x}{x+1}=\frac{3x}{x+2}-2.$

Solution: The least common multiple of the denominators is $(x+1)(x+2)$.

Hence, $\frac{\cancel{(x+1)}(x+2)x}{\cancel{x+1}}=\frac{(x+1)\cancel{(x+2)}3x}{\cancel{x+2}}-(x+1)(x+2)2$

$$\therefore x^2+2x=3x^2+3x-2x^2-6x-4.$$

28. $\frac{4}{3-x}=\frac{2}{1+x}$

29. $\frac{x+1}{x-4}=\frac{x}{x-3}$

30. $\frac{x+3}{x-2}=\frac{x+5}{x-4}.$

31. $\frac{1}{x+6}+\frac{5}{2(x+6)}=-1.$

108. What every pupil should know and be able to do. This chapter summarizes previous discussions relating to the solution of verbal problems. You should now be able to do the following:

1. To translate numerical facts stated in words into arithmetical and algebraic symbols.
2. To derive the equation which when solved gives the answers called for by the problems.
3. To solve equations in one unknown of the form $42-3x=48-9x$.
4. To solve equations containing parentheses, as $3a-2(a+5)=6a-20$.
5. To solve fractional equations which reduce to equations of the first degree.

109. Typical problems and exercises. The following exercises are of the types that you should be able to solve. Work them and others of similar types if you need the practice.

1. Solve for a and check: $4a-15-a=35-2a$.

2. Solve for y: $8(3-2y)-2(5-y)=28$.

3. Solve for a: $\frac{a+13}{13}-\frac{6-3a}{65}=1$.

4. Select one typical problem out of each of the following sets and solve: §§ 99, 100, 101, 102, 103, 105, 106.

5. Prepare a paper or a brief talk on one of the following topics. Select examples to illustrate your discussion.

a. How to derive equations from verbal problems.

b. How to solve simple equations containing one unknown number.

c. How to solve fractional equations.

CHAPTER VII

PROBLEMS LEADING TO SIMPLE EQUATIONS IN TWO UNKNOWNS

Graphical Solution

110. Problems containing several unknowns. The problem: "If one angle of a triangle is twice as large as the second, and the third is three times the second, find the angles," calls for three unknown angles. Expressed in terms of one letter they are $2x$, x, and $3x$. They may be found by solving the equation

$$2x+x+3x=180.$$

Frequently problems call for several unknown numbers. In all previous work, when we solved problems containing several unknowns, we denoted one of them by a letter and expressed the others in terms of that letter. Sometimes the task of expressing all unknowns in terms of one letter is not simple, and it saves time and effort to solve the problem by using several letters, each denoting one of the unknown numbers, as will be shown in the problem following.

Two boys were making purchases for a picnic. One bought 60 cents worth of oranges and apples, the other bought 50 cents worth. The first received 6 oranges and 10 apples, the other received 7 oranges and 5 apples of the same kind as the first. Later they found that they needed to buy more fruit, and thereupon wished to know the price of one orange and of one apple. They worked the problem as follows:

Solution: Let x be the number of cents paid for an orange.
Let y be the number of cents paid for an apple.
Then $6x+10y=60$
and $7x+\ 5y=50$

Thus the problem led to two equations containing two unknown numbers. To find the answer, the boys had to know how to solve this pair of equations.

In this chapter we shall learn several methods of solving such pairs of equations. One is called a *graphical solution* because graphs are used, the other is called an *algebraic solution*, because the values of the unknowns are found by means of algebraic processes.

111. How to use graphs in solving a pair of equations. If in the equation $7x+5y=50$ we assign to x a value, as $x=2$, the equation takes the form

$$7\cdot 2+5y=50.$$

From this we have

$$\begin{aligned} 14+5y&=50\\ 5y&=36\\ y&=7.2 \end{aligned}$$

The number pair (2, 7.2) *satisfies* the equation, for

$$(7)(2)+(5)(7.2)=14+36=50$$

For this reason the pair of numbers $(x, y)=(2, 7.2)$ is called a *solution* of the equation $7x+5y=50$.

Similarly, other solutions may be found by assigning other values to x:

$$\begin{aligned} \text{If } x=0, \text{ then } (7)(0)+5y&=50 \\ 0+5y&=50 \\ \therefore y&=10 \\ \text{If } x=1, \text{ then } (7)(1)+5y&=50 \\ 5y&=43 \\ \therefore y&=8.6 \\ \text{If } x=3, \text{ then } (7)(3)+5y&=50 \\ 5y&=29 \\ \therefore y&=5.8. \end{aligned}$$

In a similar way verify the correctness of the other pairs of values tabulated in Table 2 (Fig. 116).

Note that the equation has an *unlimited* number of solutions.

The number pairs in table (2) may be represented graphically as follows:

Draw two reference axes, OX and OY (Fig. 116). Values of x are laid off on the x-axis and values of y are laid off at right angles to the x-axis. Thus, to plot the pair (2, 7.2) pass from 0 two units to the right and then 7.2 units upward, locating point A. When all number pairs given in the table have been plotted as points, draw the line AB passing through these points. This line is said to be the graph of the equation $7x+5y=50$.

Verify the following two characteristics of this line:

1. A solution of the equation, when plotted, locates a point on the line.

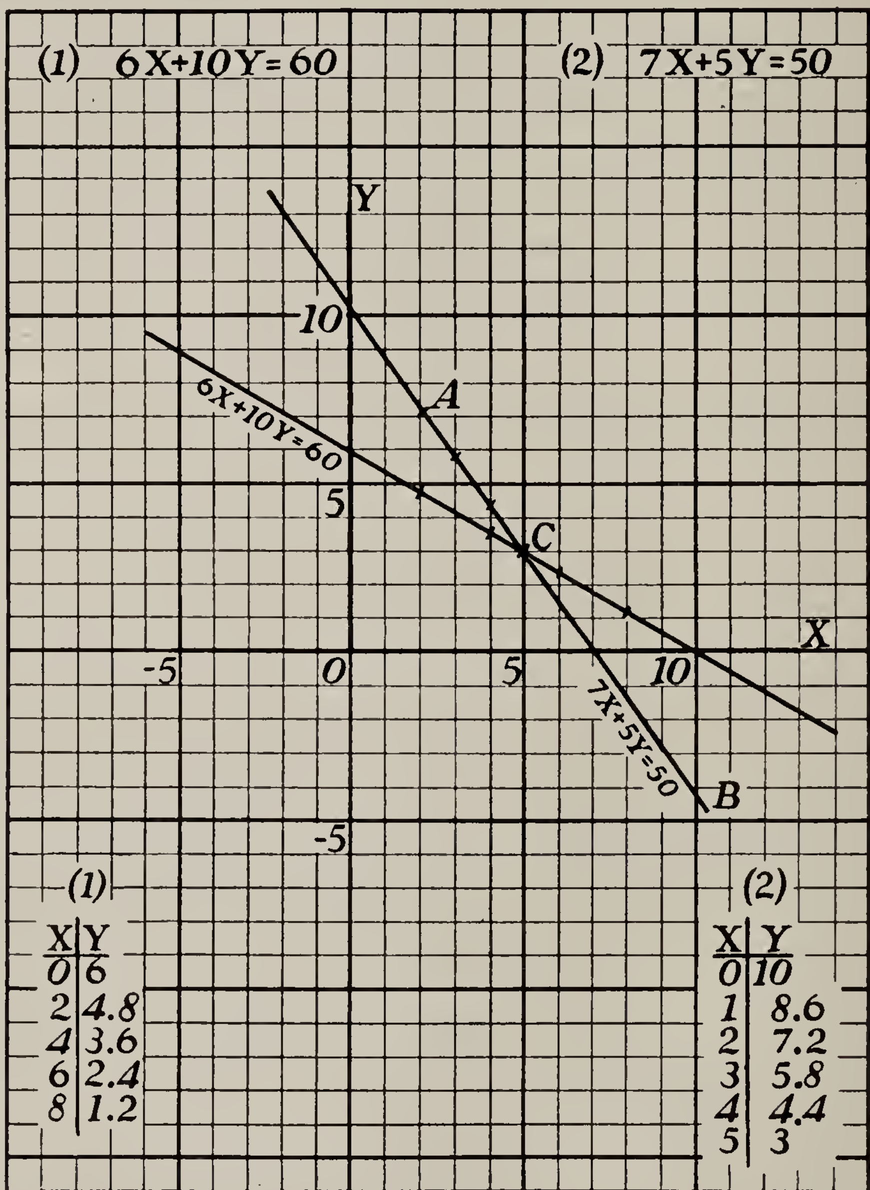


Fig. 116

2. Any point on the line determines a pair of numbers satisfying the equation.

For example, for the point B we have $x=10$ and $y=-4$. Substituting in the equation, we have

$$(7)(10)+(5)(-4)=70-20=50,$$

which shows that the equation is satisfied.

Because the graph is a straight line, equations like $7x+5y=50$ are called *linear* equations. A pair of such equations is called a *system* of equations.

Verify table (1), which gives solutions of the equation $6x+10y=60$.

Plot the points and make the graph of the equation.

The two straight lines intersect at C. To this point corresponds the pair $(x, y)=(5, 3)$.

Since a pair of numbers corresponding to a point on a graph is a solution of the equation represented by the graph, it follows that the pair which corresponds to the point of intersection of *two* graphs is a solution of *both* equations.

Therefore the solution of the equations is the number pair $(x, y)=(5, 3)$.

This is also the solution of the original problem. Hence, the price of an orange was 5 cents, and the price of an apple was 3 cents.

112. Summary of the steps in the graphical solution of a system of equations. Let $6x+10y=60$ and $7x+5y=50$ be the system of equations to be solved. The following is a summary of the steps in the process of solving:

1. *Assume values of x and find the corresponding values of y.* Since the lines are straight lines, two or three points are sufficient to determine the line. For convenience, always

First let $x=0$. Then $(6)(0)+10y=60$, and $y=6$
Next, let $y=0$. Then $6x+(10)(0)=60$ and $x=10$.

Finally, choose a third value for x not too near the other two values, as $x=4$. Then $24+10y=60$ and $y=3.6$.

Similarly for the second equation,

Let $x=0$. Then $y=10$.
Let $y=0$. Then $x=7.1$.
Let $x=3$. Then $y=5.8$.

2. *Tabulate the number pairs*, as in tables (1) and (2) (Fig. 116).

3. *Draw the axes, select a convenient unit, and plot the number pairs given in the tables.*

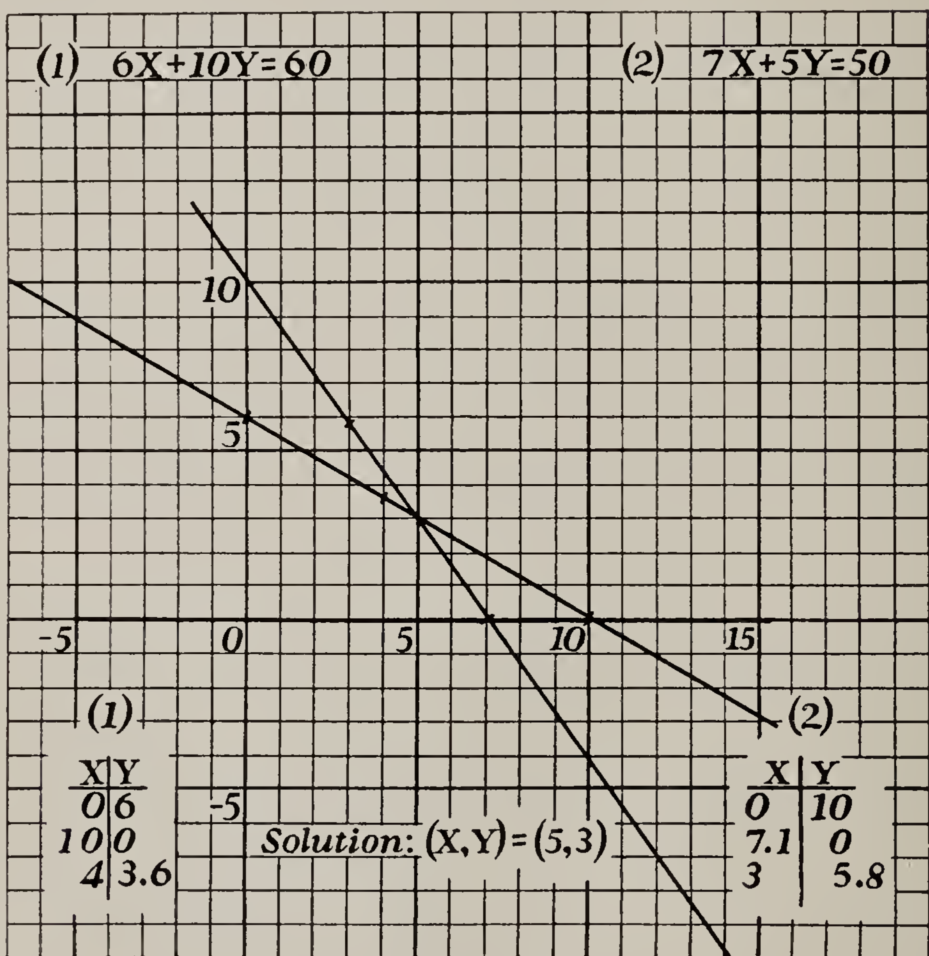


FIG. 117

4. *Draw the graphs and locate the point of intersection C.*

5. *Starting from 0 and passing along the x-axis, count the number of units to the foot of the perpendicular from C to the x-axis.* This is the required value of x.

Count the number of units contained in the perpendicular. This is the required value of y.

6. *The pair of numbers* $(x, y) = (5, 3)$ *thus determined is the required solution.*

The written work may be conveniently arranged as in Fig. 117.

Exercises

Solve the following equations graphically:

1. $x - y = 4$
   $x + y = 18.$

2. $x + y = 6$
   $2x - 3y = 2.$

3. $x + y = 5$
   $2x - 5y = -11.$

4. $6x - 5y = 15$
   $3x + 2y = 21.$

5. $x + y = 6$
   $3x - 2y = -2.$

6. $2x + 3y = 4$
   $5x - 2y = -9.$

7. $3x - 2y = -1$
   $2x + y = 11.$

8. $3x - 2y = 8$
   $4x - 3y = 10.$

Algebraic Solution of Equations in Two Unknowns

113. How to solve a pair of equations by algebraic methods. The graphical method explained in §§111, 112 has several disadvantages. The process is long, and it is difficult to determine accurate solutions when the unknown numbers are not whole numbers. The algebraic method avoids both of these difficulties.

Let it be required to solve the system

$$\begin{aligned} 5x + 2y &= 34 \\ 7x - 3y &= 7 \end{aligned}$$

Multiply all terms of the first equation by 7 and of the second by 5. Then

$$\begin{array}{r|l} \text{we have } 7 & 5x+2y=34 \\ \text{and } 5 & 7x-3y=\ \ 7 \end{array}$$

$$\begin{array}{rl} \text{or} & 35x+14y=238 \\ \text{and} & 35x-15y=\ \ 35 \end{array}$$

By subtracting the last equation from the one above we *eliminate* (remove) x. This gives

$$29y=203$$
$$\therefore y=\ \ 7.$$

Instead of eliminating x, we might have eliminated y by multiplying the first equation by 3, the second by 2, and adding the resulting equations.

Having determined the value of one unknown (x, or y), we may substitute it in one of the given equations. Thus, when $y=7$, the first equation changes to

$$5x+2\cdot 7=34$$
$$5x+14=34$$
$$\therefore x=4.$$

Hence the solution is $(x, y)=(4, 7)$.

The check consists in verifying *both* of the given equations.

Check: Left Member				Right Member
5×4	$+$	2×7		34
20	$+$	14		34
		34	$=$	34
7×4	$-$	3×7		7
28	$-$	21		7
		7	$=$	7

Exercises

Solve the following systems algebraically:

1. $3x-7y=40$
$4x-3y=9$

2. $9a-4b=3$
$7a+2b=33$

3. $3x-2y=8$
$4x-3y=10$

4. $2x-3y=-11$
$5x-2y=6$

5. $3a-13b=41$
$8a+11b=18$

6. $13x+3y=14$
$7x-2y=22$

7. $5a+2b=36$
$2a+3b=43$

8. $2x+3y=27$
$5x-2y=1$

9. $3m+2n=23$
$2m+3n=27$

10. $x+y=12$
$x-y=4$

11. $x+3y=7$
$x-2y=2$

12. $x+2y=4$
$3x-2y=2$

13. $3x-2y=-1$
$2x+y=11$

14. $x+4y=4$
$x-2y=16$

15. $x=2y$
$3x-y=15$

Suggestion: Eliminate x by substituting $2y$ for x in the second equation.

16. $3x=y$
$15x-4y=27$

17. $5x=36-2y$
$2x+3y-43=0$

18. $2x+6y-28=0$
$2x-y-7=0$

114. Problems leading to equations in two unknowns. The following problems lead to two equations in two unknowns. Solve them by the algebraic method.

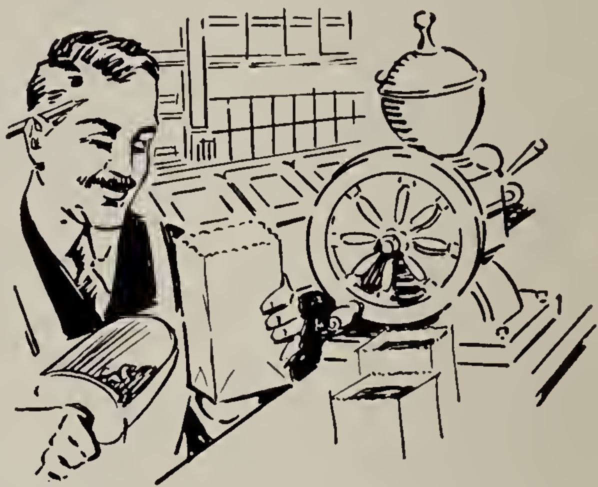
Exercises

1. Two pencils and 3 tablets cost 24 cents. Three pencils and 5 tablets cost 39 cents. Find the cost of each.

2. A boy bought 8 apples and 6 oranges for 98 cents. His sister paid 82 cents for enough to make a dozen of each. What was the price of one orange and of one apple?

3. Walnuts selling at 35 cents a pound are to be mixed with other nuts selling at 25 cents so that a mixture of 8 pounds can be sold for \$2.20. How many pounds of each are to be used?

4. A grocer wishes to mix 5 pounds of coffee, worth 39 cents per pound, by mixing two other grades, one worth 48 cents and the other 36 cents per pound. How much of each must he use?



5. Tea selling at 40 cents a pound is to be mixed with a grade selling at 60 cents a pound. How many pounds of each must be used to make 60 pounds selling at 54 cents a pound?

6. A boy can row 5 miles downstream in 1 hour and return in $1\frac{2}{3}$ hours. What is his rate of rowing in still water, and what is the rate of the current?

Suggestion: Let x be the rate of rowing in still water, and y the rate of the current.

Then $x+y$ is the rate going downstream, and $x-y$ is the rate returning.

7. A man rows 10 miles downstream in 2 hours. Find the rate of the current and the rate of rowing in still water if it takes him 2 hours and 30 minutes to return.

Solution:

	Going Down	*Returning*
Distance	10	10
Time	2	$2\frac{1}{2}$
Rate	5	$\frac{10}{2\frac{1}{2}}$

$$\therefore x+y=5$$

$$\text{and } x-y=\frac{10}{2\frac{1}{2}}.$$

Solve this system of equations.

8. In 50 minutes a man rows 50 miles downstream and in 1 hour and 30 minutes 12 miles upstream. Find the rate of the current and how fast he can row in still water.

9. The larger of two numbers is 17 greater than 4 times the smaller, and twice the larger number is 48 greater than 7 times the smaller. Find the two numbers.

10. The sum of $6000 is to be divided among three partners so that the first receives twice as much as the second, and the third as much as the first two together. How much does each receive?

11. A grocer wishes to make a 5-pound mixture of coffee selling for $2.20 from two kinds of coffee, one selling at 40 and the other at 50 cents a pound. How much of each kind must he use?

Solution: The unknown and the given facts of this problem are as follows:

Let x be the number of pounds in the 40 cent mixture
Let y be the number of pounds in the 50 cent mixture
5 is the number of pounds in the required mixture

40 is the price per pound of the first mixture
and 50 is the price per pound of the second mixture
220 is the total price of the required mixture

$\therefore$ $40x$ is the total price of the first mixture
and $50y$ is the total price of the second mixture.

To avoid repetition and to make the facts of the problem stand out clearly, it is convenient to arrange them in tabular form as follows:

	40c Mixture	*50c Mixture*	*Required Mixture*
Number of pounds	x	y	5
Price per pound	40	50	
Total price	$40x$	$50y$	220

Show that the equations are:

$x+y=5$, expressing the number of pounds in the required mixture.
$40x+50y=220$, expressing the total price of the required mixture.
Solve this system of equations.

12. A grocer wishes to make a mixture from two kinds of coffee, one selling for 35 cents a pound and the other for 25 cents a pound. The mixture is to contain twice as much of the first coffee as of the second and is to sell for $9.50. How much of each is to be used?

13. A farmer has two qualities of milk, one 3% and the other 5%. How much of each shall he use to make 10 gallons of 4.5% milk?

14. A farmer wants to send an order for 48 bags of cement to a wholesale house and is puzzled to know how much money he must enclose. He has no catalog but his son remembers that in making a cement walk costing $26 they had used 8 bags of cement and 2 cubic yards of sand, and in making a reservoir for water they had used 24 bags of cement and 5 cubic yards of sand at a total cost of $73. Can you determine how much money to send?

15. A man invests two sums at 5 per cent and at 4 per cent respectively, and from this investment he has a yearly income of $500. If the total sum invested amounted to $12,000, how much did he invest at each rate?

16. A man invests part of $10,000 at 6 per cent interest and the remainder at 5 per cent. The total yearly income is $570. Find the amount invested at each rate.

17. A man has a yearly income of $54 from an investment of $1000. If one part of the investment yields 6 per cent and the remainder 5 per cent, find the amount invested at each rate.

115. What every pupil should know and be able to do. It is expected that at the end of Chapter VII you should be able to do the following:

1. To translate the facts contained in verbal problems into algebraic symbols.
2. To derive the equation, or equations, from which the solution may be obtained.

3. To solve a pair of simple equations in two unknowns.

a. By the graphical method.

b. By eliminating one of the unknowns.

116. Typical problems and exercises. Every pupil should be able to solve the following equations and problems:

1. Using the method of §112, solve graphically the system:

$3x-2y=-1$
$2x+\ y=11.$

2. Solve by eliminating x:

$3x-7y=40$
$4x-3y=9.$

3. Solve by eliminating y:

$13x+3y=14$
$7x-2y=22.$

4. Eliminate y by substitution:

$$\begin{aligned} 3x \qquad &= y \\ 15x-4y &= 27 \end{aligned}$$

5. Select at random from pages 169 to 172 one problem in each of the following sets and solve it:

1 to 5; 6 to 8; 9 to 13; 14 to 17.

6. Prepare a talk or write a paper on one of the following topics:

a. How to solve a pair of equations graphically.

b. How to solve a pair of equations algebraically.

c. How to solve verbal problems.

CHAPTER VIII

PROBLEMS LEADING TO QUADRATIC EQUATIONS

What We Have Previously Learned about Quadratic Equations

117. Quadratic equations were found in studying areas and volumes. Our first quadratic equation appeared in §22 in the following problem: "A field of the form of a square has an area of 100 square rods. Find the length of the side." The equation for solving the problem is $a^2=100$. The method of solving is to extract the square root of both members, which gives $a=10$.

A similar equation occurred in studying the area of a circle. We have seen that the area of a circle is found from the formula $A=\pi r^2$. To find the radius when the area is 164 we have to solve the quadratic equation $\pi r^2=164$.

Solution: $\pi r^2=164$

Dividing both members by the coefficient of r^2, we have

$$r^2=\frac{164}{\pi}$$

Taking the square root of both members,

$$r=\sqrt{\frac{164}{\pi}}$$

To complete the solution, use $\pi=3.142$, find the quotient 164 to three figures, and extract the square root.

The following exercises give practice in solving problems which lead to equations similar to those explained above.

Exercises

1. Find the radius of a circle whose area is 37.5 square inches.

2. We learn in science that the number of feet, s, through which an object falls in t seconds is given by the formula $s=\frac{1}{2}gt^2$, where $g=32.16$ approximately. How long will it take a stone to fall 850 feet?

3. The volume of a cylinder is 24 cubic inches. The altitude is 6 inches. Find the height.

4. The area of the surface of a sphere is 28 square inches. Find the radius.

Solve the following equations:

5. $x^2=25$.

6. $x^2-121=0$.

7. $2x^2-30=0$.

8. $6a^2-72=0$.

9. Solve the equation $s=\frac{1}{2}gt^2$ for t.

Solution: $s=\frac{1}{2}gt^2$.

Dividing both members of the equation by the coefficient of t^2, we have

$$\frac{s}{\frac{1}{2}g}=t^2,$$

$$\text{or } \frac{2s}{g}=t^2$$

Taking the square root, we have

$$\sqrt{\frac{2s}{g}}=t.$$

10. Using the result of Exercise 9 as a formula, find t when $s=8$; 24; 16; 48.

11. The area of an equilateral triangle is given by the formula $A=\frac{a^2}{4}\sqrt{3}$, where a is the length of the side. Find the side of an equilateral triangle whose area is 620 square feet.

12. Solve the equation $A=\frac{a^2}{4}\sqrt{3}$ for a, using the method explained in Exercise 9.

13. The volume of a right circular cylinder is given by the formula $V=\pi r^2 h$. Solve the equation for r.

14. The area of the surface of a sphere is found from the formula $S=4\pi r^2$. Solve the equation for r.

15. Solve the equation $f=\frac{mv^2}{r}$ for v.

118. The relation between the sides of the right triangle. The following problem shows how we use the quadratic equation when it is required to find a side of a right triangle having given the other two sides.

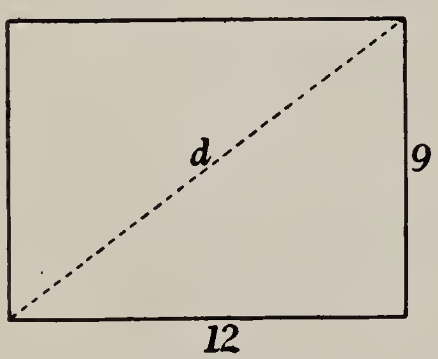


Fig. 118

The dimensions of a room are 12 feet and 9 feet. How far apart are two diagonally opposite corners?

Solution: Let d be the number of feet in the diagonal (Fig. 118).

Then by the theorem of Pythagoras we have

$$d^2=9^2+12^2=81+144=225$$
$$\therefore d^2=225$$
$$\therefore d=15.$$

Exercises

1. One side of a right triangle is 18 inches and the hypotenuse is 30 inches long. Find the length of the remaining side.

2. If the equal sides of an isosceles triangle are 60 inches and the altitude 36 inches, find the base.

Suggestion: Draw the altitude dividing the given triangle into two right triangles. Use the theorem of Pythagoras with the sides of one of the right triangles.

Graphical Solution of Quadratic Equations

119. Equation of motion of an object thrown vertically upward. When a ball is thrown upward, its velocity decreases. It becomes zero when the greatest height is reached. Then, as the ball drops back, the velocity increases. The greatest height reached by a ball thrown upward depends upon the velocity with which the ball is thrown. Experiments in science have shown that the ball rises and falls according to a law which can be expressed mathematically. If we denote by v the velocity with which the ball is thrown, and by t the number of seconds in which it reaches a height d, then d can be found from the formula $d = vt - \frac{gt^2}{2}$.

The number g is approximately 32.16. It denotes the velocity acquired by an object falling unresisted for one second under the action of gravity.

The equation shows that the height d depends on the time t and the initial velocity v. If we substitute

for g the value 32.16, the equation changes to the form: $d = vt - 16.08t^2$. The equation may be used (1) to determine the height d for a given value of t, and (2) to find the time t in which the ball rises to any given height d.

Suppose a ball is thrown vertically upward with a velocity of 100 feet a second.

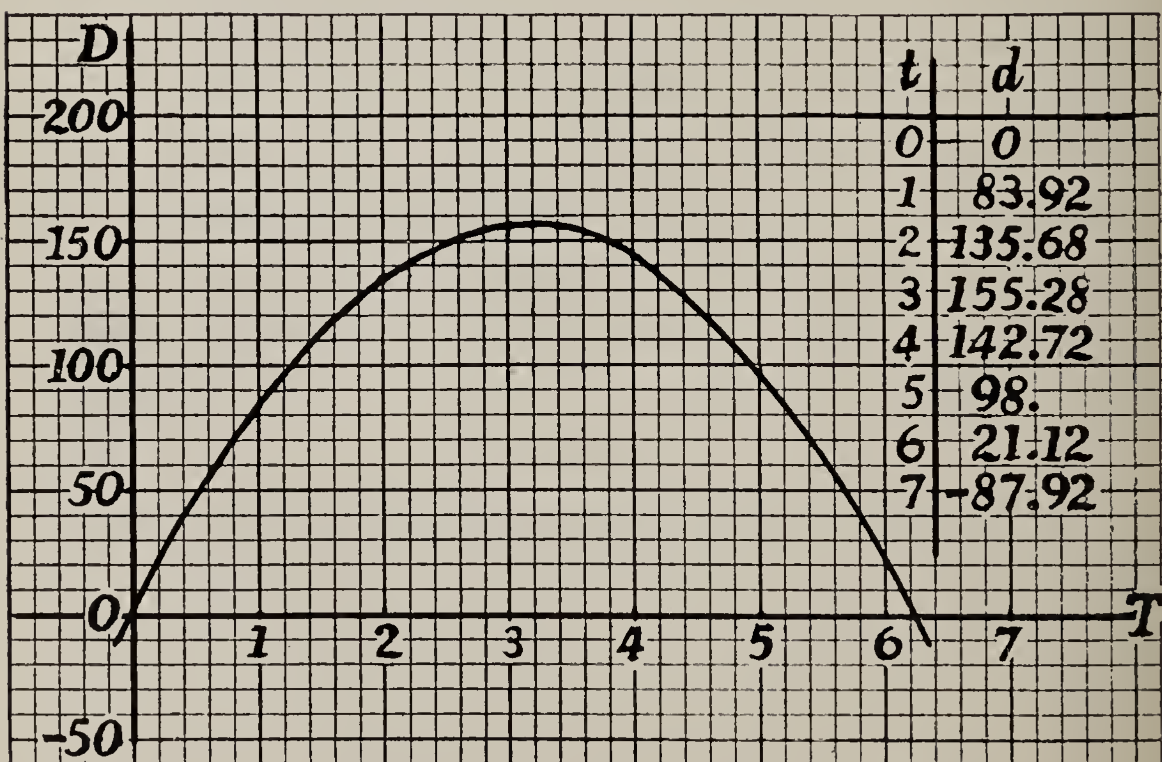
$$\text{Then } d = 100t - 16.08t^2$$

Let $t = 1, 2, 3, 4$, etc., and find corresponding values of d.

Tabulate the corresponding values of d and t (Fig. 119).

Plot the number pairs in the table and draw the graph.

From the graph we are able to determine approximate values of t for given values of d.



t	d
0	0
1	83.92
2	135.68
3	155.28
4	142.72
5	98.
6	21.12
7	-87.92

FIG. 119

Exercises

1. Using the graph (Fig. 119) answer questions a, b, and c below:

a. When will the ball be 50 feet from the ground?
b. When will the ball be 75 feet from the ground?
c. When will the ball be 125 feet from the ground?

2. By means of the graph find t when $d = 140$.

3. Show that the two values of t found in Exercise 2 are the solutions of the equation $140 = 100t - 16.08t^2$.

120. Quadratic equations. An equation like

$$16.08\,t^2 - 100\,t + 140 = 0 \text{ (Exercise 3)}$$

is of the *second degree* in t, because it *contains the second power of t and no higher power.* It is a *quadratic* equation. Show similarly that the equations $x^2 - 5x + 6 = 0$; $2x^2 - 3x = 0$; $5x^2 = 16$ are quadratic equations.

121. How to use the graph to solve a quadratic equation. The exercises of §119 indicate a way of solving a quadratic equation graphically. The method may be summarized in the following steps:

1. *Bring all the terms to one side of the equation, making the other side equal to zero.* For example, the equations $4x^2 - 4x = 15$ and $4x^2 = 4x + 15$ are changed to $4x^2 - 4x - 15 = 0$.

2. *Tabulate corresponding pairs of values of x and* $4x^2 - 4x - 15$.

Thus, if $x = 0$, $4x^2 - 4x - 15 = -15$
if $x = 1$, $4x^2 - 4x - 15 = 4 - 4 - 15 = -15$
if $x = 2$, $4x^2 - 4x - 15 = 16 - 8 - 15 = -7$
if $x = 3$, $4x^2 - 4x - 15 = 36 - 12 - 15 = 9$, etc.

3. *Plot these pairs, and make the graph by drawing a smooth curved line through the points* (Fig. 120).

4. *Measure the distances from the origin to the points of intersection of the curve with the x-axis.* These are the values of the unknown, for which $4x^2-4x-15$ is equal to zero. If we denote these values by x_1 and x_2, it follows that $x_1=-1.5$ and $x_2=2.5$ are the required solutions.

The numbers -1.5 and 2.5 are called the *roots* of the quadratic equation $4x^2-4x-15=0$. Note that a quadratic equation has *two* roots corresponding to the two points of intersection of the curve with the x-axis.

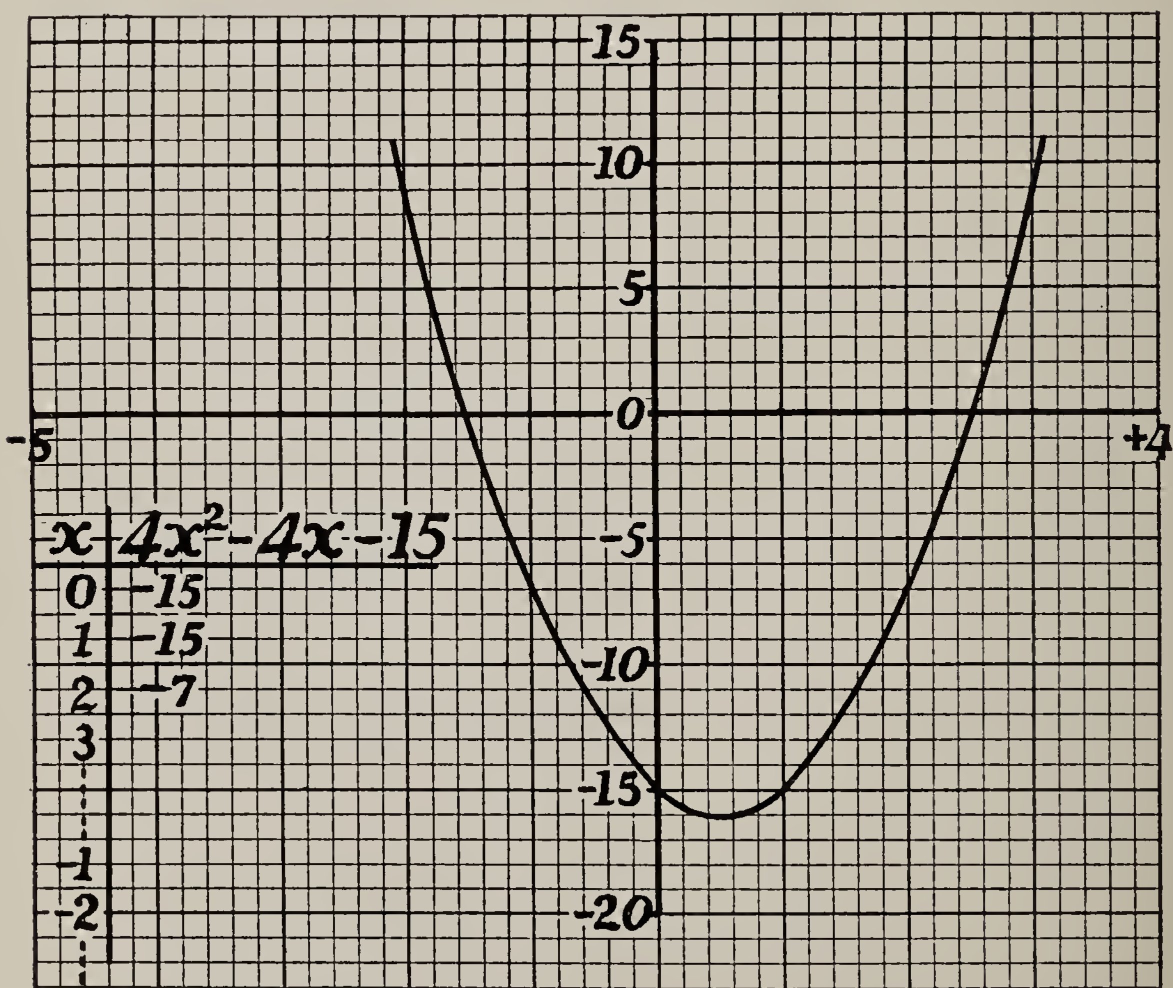


FIG. 120

Exercises

Solve the following quadratic equations graphically:

1. $m^2-4m-12=0$.
2. $y^2+y=56$.
3. $y^2=4y-3$.
4. $x^2-8x+12=0$.
5. $4x^2-12x+5=0$.
6. $x^2-34x+145=0$.

Algebraic Solution of Quadratic Equations

122. Every quadratic equation has two roots. The graphical solution of the quadratic equation (Fig. 120) shows that the equation $4x^2-4x-15=0$ has *two* roots. This is true of every quadratic equation. For example, the equation $x^2-25=0$ is satisfied by *two* values of x, *i.e.*, $+5$ and -5. This is easily verified by substituting in the equation the numbers $+5$ and -5 in place of x.

The complete solution of the equation $x^2-25=0$ may now be arranged as follows:

$$\begin{aligned} x^2-25&=0 \\ x^2&=25 \\ \sqrt{x^2}&=\pm\sqrt{25} \text{ (read: + or − square root of 25)} \\ x_1&=+5 \\ x_2&=-5 \end{aligned}$$

By arithmetic we were able to find only one square root of a number. We have just seen that there is always a second square root. Thus, the square roots of 36 are $+6$ and -6, because $(+6)^2=36$ and $(-6)^2=36$.

Exercises

Solve the following equations as shown in Exercise 1:

1. $(x+1)^2=16.$

Solution: Extracting the square root we have

$$x+1=\pm 4$$
$$\therefore x+1=+4,$$
$$\text{and } x+1=-4$$
$$\therefore \; x_1=3,$$
$$\text{and } x_2=-5.$$

2. $(x+3)^2=4.$

3. $(x-8)^2=25.$

4. $(x+2)^2=16.$

5. $(x-5)^2=26.$

6. $(x-4)^2=18.$

7. $(y+6)^2=10.$

8. $(a+8)^2=12.$

9. $(m-11)^2=20.$

123. Solving quadratic equations in which one member is a quadratic trinomial square. The quadratic equations in §117 were all of a simple form, the first degree term being missing. Before taking up the solution of the complete equation, let us consider further equations which reduce to the form shown in Exercises 1 to 5 above.

Squaring both members of the equation

$$x+1=4$$

$$\text{we have } (x+1)^2=16 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$\text{or } x^2+2x+1=16 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Equations like (2), in which the left member is a perfect square, can always be solved by *first changing them to form* (1), and then using the method shown in Exercise 1 (§ 122).

EXERCISES

Exercises 1 and 2 below show how to change a trinomial square into the square of a binomial.

1. Change x^2+2x+1 into a square of a binomial.

Solution: Show by multiplying $x+1$ by $x+1$ that $x^2+2x+1=(x+1)^2$. The binomial $x+1$ may be obtained from x^2+2x+1 by inspection as follows:

a. Extract the square root of the first term, x^2.
b. Extract the square root of the third term, 1.
c. *Add* the results, which gives $x+1$.

2. Change $x^2-14x+49$ into a square of a binomial.

Solution: Extract the square root of the first term, x^2.
Extract the square root of the third term, 49.
State the *difference* of the results, which gives $x-7$.
Test the answer by multiplying $x-7$ by $x-7$.

Change each of the following trinomials into a square of a binomial. Then verify the result by multiplying the binomial by itself.

3. x^2+6x+9.
4. x^2+4x+4.
5. $y^2+8y+16$.
6. $m^2-12m+36$.
7. $a^2-10a+25$.
8. r^2-4r+4.

The left members of the quadratic equations below are perfect trinomial squares. Solve each as shown in Exercise 9.

9. $x^2+8x+16=9$.

Solution: $x^2+8x+16=9$

Change the left member into a square of a binomial, by taking the square root of the first term and the square root of the third term and adding the results. Thus we have

$$(x+4)^2=9.$$

Extracting the square root, we find that

$$x+4=\pm 3 \therefore$$

$$x_1=-1 \text{ and}$$

$$x_2=-7.$$

Check: LEFT MEMBER RIGHT MEMBER

LEFT MEMBER		RIGHT MEMBER
$x^2+8x+16$		9
$(-1)^2+8(-1)+16$		9
$1-8+16$		9
9	=	9
$(-7)^2+8(-7)+16$		9
$49-56+16$		9
9	=	9

10. $x^2+6x+9=16.$

11. $x^2-4x+4=25.$

12. $x^2+8x+16=49.$

13. $x^2-x+\frac{1}{4}=9.$

14. $x^2+8x+16=7.$

15. $x^2-10x+25=11.$

16. $x^2+\frac{2x}{3}+\frac{1}{9}=16.$

17. $x^2-\frac{x}{2}+\frac{1}{16}=25.$

124. Solving a quadratic equation by completing the square. In §123 we have learned how to change a perfect trinomial square into a square of a binomial. Thus, x^2+6x+9 may be changed to $(x+3)^2$. Geometrically this means that the whole square $(x+3)^2$ (Fig. 121) is equal to the sum of the parts x^2, $3x$, $3x$, 9. The third term, 9, of the trinomial (x^2+6x+9) is related to the linear term, $6x$, and may be found from it by *taking one-half of the coefficient of x and then squaring it.*

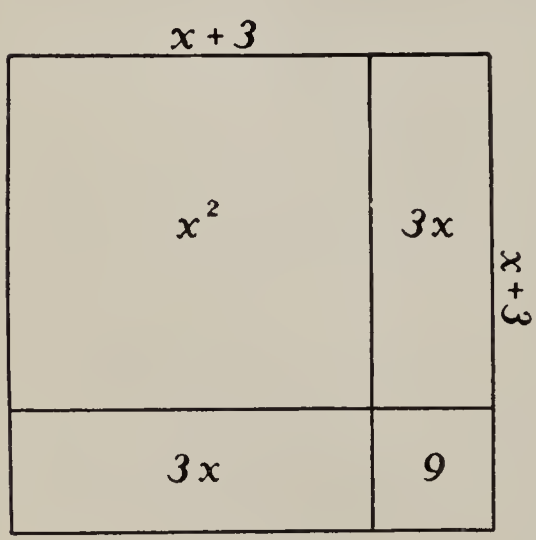


FIG. 121

Similarly, to form the square whose first two terms are x^2+16x, take $\frac{1}{2}$ of 16 and square the result, which

gives 64. Add 64 to x^2+16x and the result $x^2+16x+64$ is a perfect square.

We are now able to solve *any* quadratic equation containing one unknown. The solution is illustrated in the following example. Let it be required to solve the equation

$$x^2+6x-55=0.$$

Add 55 to both members. This gives

$$x^2+6x=55.$$

Find one-half of the coefficient of x in $6x$ and square it. The result is 9.

Complete the square on the left side by adding 9 to both members:

$$x^2+6x+9=64$$

Change the left member to a square of a binomial:

$$(x+3)^2=64$$

Extract the square root:

$$x+3=\pm 8$$

Solving for x, we have $x_1=5$

$$x_2=-11$$

State the five steps in the solution of the preceding equation.

Exercises

To each of the binomials in Exercises 1 to 9 add the term needed to make a complete trinomial square:

1. x^2+6x.
2. x^2-8x.
3. x^2+4x.
4. x^2-3x.
5. x^2+x.
6. x^2+7x.
7. $x^2-\frac{1}{2}x$.
8. $x^2+\frac{1}{3}x$.
9. $x^2+\frac{2}{3}x$.

Solve the following equations in Exercises 10 to 19 by completing the square:

10. $x^2+3x+2=0$.

11. $x^2-8x-48=0$.

12. $a^2+8a-20=0$.

13. $y^2+4y+3=0$.

14. $x^2+6x-16=0$.

15. $x^2+4x=21$.

16. $x^2+6x+5=0$.

17. $x^2+4x-32=0$

18. $3+2x=x^2$.

19. $x^2=3x+4$.

The equations in Exercises 20 to 28 differ from those above in the coefficient of x^2. To solve these equations divide first by the coefficient of x^2 and then proceed as in Exercises 10 to 19.

20. $3x^2-2x-3=0$.

Solution: Divide every term by 3.

$$\text{Then } x^2-\frac{2x}{3}-1=0$$

Add 1 to both members: $x^2-\dfrac{2x}{3}=1$

Add the square of $\frac{1}{2}$ of $-\frac{2}{3}$:

$$x^2-\frac{2x}{3}+\tfrac{1}{9}=1+\tfrac{1}{9}$$

$$\therefore (x-\tfrac{1}{3})^2=\tfrac{10}{9}$$

$$\therefore x-\tfrac{1}{3}=\frac{\pm\sqrt{10}}{3}$$

Solve for x: $x=\frac{1}{3}\pm\dfrac{\sqrt{10}}{3}=\dfrac{1\pm\sqrt{10}}{3}$

State the steps in the solution of the equation above.

21. $3x^2-7x-20=0$.

22. $4x^2-4x-79=0$.

23. $3a^2-7a=6$.

24. $4y^2=1-4y$.

25. $x^2+18x-15=0$.

26. $5a^2+25a=-9$.

27. $6x^2=3x+45$.

28. $2x(x+4)=42$.

125. Problems solved by means of quadratic equations. The following problems lead to quadratic equations which may be solved by the methods explained above:

EXERCISES

1. The sum of two numbers is 42 and the product is 416. Find the numbers.

2. The sum of the squares of two consecutive numbers is 421. Find the numbers.

Suggestion: Let x be one number. Then $x+1$ is the consecutive number.

3. The sum of the squares of two consecutive numbers is 1013. What are the numbers?

4. Three times the square of a number if increased by the number is equal to 16. Find the number.

5. The perimeter of a rectangle is 84 rods and the area is 432 square rods. Find the dimensions.

6. The length of a rectangle exceeds the breadth by 4 inches. The area is 140 square inches. Find the dimensions.

7. The base of a triangle exceeds twice the altitude by 4 inches, and the area is 63 square inches. Find the base and altitude.

8. The base of a triangle exceeds the altitude by 4 inches. The area is 30 square inches. Find the base and altitude.

9. The hypotenuse of a right triangle is 9 feet longer than one of the other sides and 2 feet longer than the third side. Find the three sides of the triangle.

10. The hypotenuse of a right triangle is 10 inches and the sum of the other two sides is 14 inches. Find the lengths of the sides.

11. The sum of the areas of two squares is 61 square rods. A side of one is 1 rod longer than a side of the other. Find the sides of the squares.

12. Find the side of a square whose area is doubled if the dimensions are increased by 9 feet and 6 feet respectively.

13. Find the dimensions of a coal bin holding 6 tons of coal whose depth is 6 feet. The length is equal to the sum of the width and depth and one ton of coal takes up 32 cubic feet of space.

14. An automobile travels 160 miles. Returning, it increases its speed by 4 miles an hour. Find the rate if the round trip takes 9 hours.

15. Two men starting from the same place and at the same time walk at rates of 3 and 4 miles an hour respectively. If the first walks east and the other north, how soon will they be 15 miles apart?

16. The radius of one circle is 7 inches longer than that of another, and the area of the first is 770 square inches greater than that of the second. Find the radii.

17. By lengthening the radius of a sphere by 2 feet, we double its surface. Find the original radius to 2 figures.

18. Proportion in design means a relationship between measures of different parts of a whole. For example, in a rectangle it might mean a relation between the measures of two adjacent sides. Much effort has been devoted in the search for guiding principles in design practice. The approximate ratio of two parts to three has been used widely and successfully. Another ratio is determined as follows:

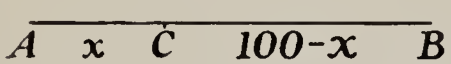


Fig. 122

Let point C (Fig. 122) be determined so that

$$\frac{AC}{CB}=\frac{CB}{AB}$$

This division of AB is known in geometry as *division in mean and extreme ratio*, or as the *golden section*. Thus, if AB be divided into 100 equal parts, we have

$$\frac{x}{100-x}=\frac{100-x}{100}$$

Multiplying both members of this equation by $100(100-x)$

we have $$100x=100^2-200x+x^2$$

or $$x^2-300x+10000=0$$

Solve this equation, and show that the approximate ratio of

$$\frac{AC}{CB} \text{ is } \tfrac{38}{62} = .61.$$

Since the ratio $\frac{5}{8}$ is approximately .62, designers have found it very helpful to use the ratio $\frac{5}{8}$ upon the large proportions of a design and in making subdivisions. Thus, in the rectangle $PQRS$ (Fig. 123) the dimensions are in the ratio $\frac{5}{8}$ and the diagonal is used to subdivide the rectangle, *i.e.*, to locate corners of other rectangles of the same proportions as $PQRS$.

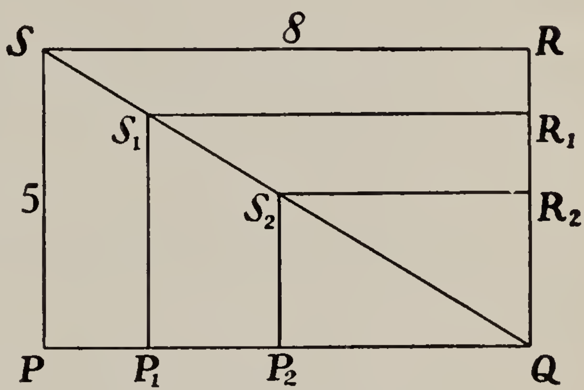


FIG. 123

19. An open box is to be made from a square piece of tin (Fig. 124) by cutting out a square from each corner and turning up the sides. If the box is to contain 180 cubic inches and is to be 5 inches high, how large a square of tin is to be used?

Suggestion: The facts of the problem can be expressed by means of a quadratic equation. This equation will be found to have two roots. Both roots are not necessarily answers to the problem. A root of the equation which does not satisfy the conditions of the problem is to be discarded.

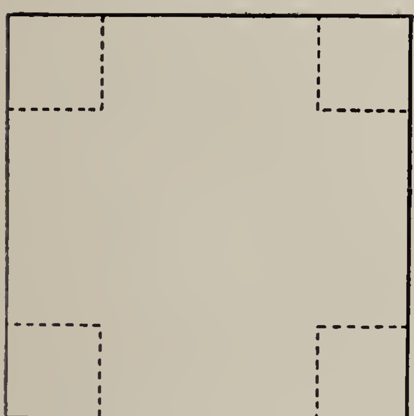


FIG. 124

20. From each corner of a square piece of tin a square is cut whose side is 8 inches. By turning up the sides a box is formed containing 1152 cubic inches. Find the length of the side of the square piece of tin.

21. In using an old plan for a new house an architect must reduce it in size. The original plan is made for a piece of ground 30 feet wide and 40 feet long. If the new house is to cover a piece of ground containing 925 square feet, by what amount must the dimensions of the plan be reduced?

22. A rectangle is twice as long as it is wide. If it were 20 rods longer and 24 rods wider, the area would be doubled. What are the dimensions?

126. What every pupil should be able to do. It is expected that you should be able to do the following:

1. To solve equations of the forms $ax^2=b$, where a and b are arithmetical numbers.

2. To solve the equation $ax^2+bx+c=0$, where a, b, and c may have any given values, using either the graphical method or the method by completing the square.

3. To solve verbal problems leading to quadratic equations.

127. Typical exercises. The following exercises indicate types of problems you should be able to do:

1. Solve for r: $v=\pi r^2h$.

2. Solve for x by means of the graph: $x^2-5x-6=0$.

3. Solve for y by completing the square: $y^2+6y-16=0$.

4. Solve several problems of the list of problems given in §125.

5. Prepare a talk or paper on one of the following topics:

a. How to solve a quadratic equation graphically.

b. How to solve a quadratic equation by completing the square.

CHAPTER IX

COMMUNITY ARITHMETIC

Taxes

128. Why taxes are necessary. The graduating class of our school decided by vote to have a class picnic. A committee has been appointed to make all arrangements for transportation, entertainment, and refreshments. The committee has made a study of all expenses and found out exactly how much money would be needed. The total sum is $102.50. Since all members of the class have the privilege of enjoying the outing, each has been asked to pay his share of the expense. There are 82 pupils in the class. How much will each pupil have to pay?

People living in the same community enjoy many privileges which individuals living alone cannot secure. In a city we find public schools, libraries, parks, and good roads. We have police and fire departments to protect us against lawlessness and fires and a health department to see that sanitary conditions prevail. All of this costs the government a great deal of money. Teachers, firemen, policemen, and librarians must receive salaries for their services. Good streets, parks, school buildings, courthouses, and city halls cost money. Since all the people who live in the city have the benefit of the comforts and conveniences provided by the

Tax Commission Form No. 2 (Twp. Org.)

SEE ASSESSOR'S NOTICE ON THE BACK OF THIS SHEET

Before commencing to fill out this Schedule, read carefully the INSTRUCTIONS TO PERSONS LISTING, and the EXTRACTS from the REVENUE LAW printed on the reverse of this sheet.

USE THIS COLUMN

A schedule of numbers, amounts, quantity and quality of all personal property in the possession or under the control of .... belonging to .... on the first day of Ap[illegible]25, as listed by .... of the Town of .... School District N[illegible]............ High School District No. .... City (or Village) of .... County of .... and State of Illinois.

To be Filled by Person or Persons Required to List Personal Property				4	5	6
1	2	3				
No.	Full Fair Cash Value	Quantity and Quality; Description, Memoranda as to Quality, Face Value, Etc.	Schedule No.	ITEMS OF PROPERTY	Full Value (as determined by Assessor)	Assessed Value (Fixed by Assessor)
			1	Horses of all ages		
			2	Cattle of all ages		
			3	Mules and Asses of all ages		
			4	Sheep of all ages		
			5	Hogs of all ages		
			6	Steam Engines, including Boilers		
			7	Fire or Burglar-Proof Safes		
			8	Billiard, Pigeon-Hole, Bagatelle, or other similar tables		
			9	Carriages and Wagons		
			9-A	Automobiles and Automobile Trucks		
			10	Watches and Clocks		
			11	Sewing and Knitting Machines		
			12	Piano Fortes		

government, they must be willing to pay for them. Each person will have to pay his share of the expenses. We say that the people pay *taxes* in return for the conveniences received from the city government. If they want more and better schools, parks, and streets, they must expect to pay more taxes.

Exercises

1. Name some things which a city government provides and which have not been mentioned above.

2. Tell why taxes are necessary and why they differ for various localities.

3. List the things which the state and national governments supply and pay for, such as harbors, canals, courts, asylums, prisons, universities, highways, army and navy.

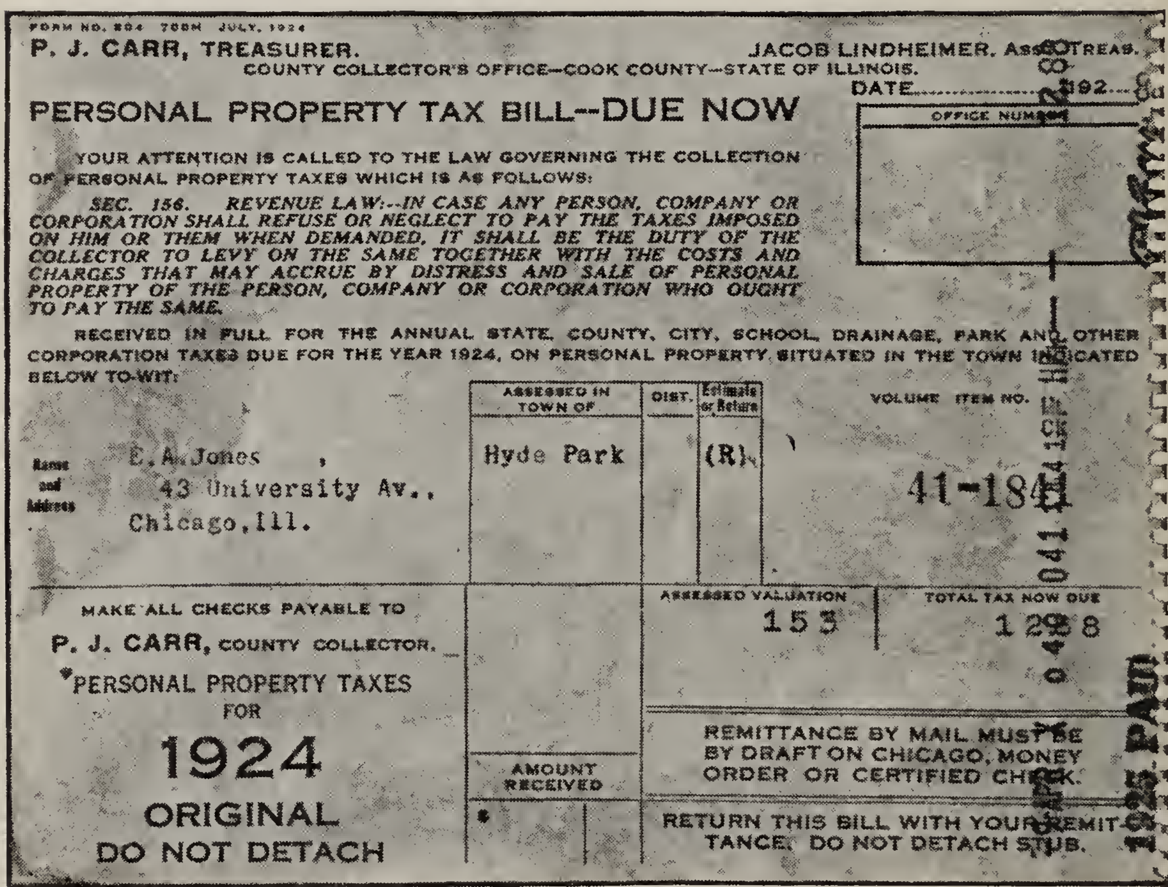
4. John lives with his parents in a little cottage in a small village. They have no telephone, gas, electric light, city water, street cars, sidewalks, or even a paved street. William's parents also have a little cottage. It is located near the city. They enjoy all the advantages which life in the village may offer, and also the conveniences just mentioned and many others. William's father pays $85 taxes each year more than John's father, but he says it costs less to live in the city than in the village where taxes amount to almost nothing. State some of the probable advantages which William derives from city life.

5. Make a list of taxes people pay for special privileges such as keeping a dog or an automobile, conducting a business, etc.

129. How the government secures the money to pay expenses. Not all people have to pay taxes. Those who own a certain amount of personal property (household goods, clothing, jewelry, horses, cattle, automobiles), and those who own real estate (land and build-

ings) pay for those who have no property. A government official known as the *assessor*, or someone from his office, makes a list of all personal property and real estate.

The value of personal property is as a rule determined by the owner, who fills out a schedule and then swears to its accuracy.


FORM NO. 804 700M JULY, 1924
P. J. CARR, TREASURER. JACOB LINDHEIMER, ASST. TREAS.
COUNTY COLLECTOR'S OFFICE—COOK COUNTY—STATE OF ILLINOIS.
DATE .... 192...

PERSONAL PROPERTY TAX BILL—DUE NOW

OFFICE NUMBER

YOUR ATTENTION IS CALLED TO THE LAW GOVERNING THE COLLECTION OF PERSONAL PROPERTY TAXES WHICH IS AS FOLLOWS:

SEC. 156. REVENUE LAW:—IN CASE ANY PERSON, COMPANY OR CORPORATION SHALL REFUSE OR NEGLECT TO PAY THE TAXES IMPOSED ON HIM OR THEM WHEN DEMANDED, IT SHALL BE THE DUTY OF THE COLLECTOR TO LEVY ON THE SAME TOGETHER WITH THE COSTS AND CHARGES THAT MAY ACCRUE BY DISTRESS AND SALE OF PERSONAL PROPERTY OF THE PERSON, COMPANY OR CORPORATION WHO OUGHT TO PAY THE SAME.

RECEIVED IN FULL FOR THE ANNUAL STATE, COUNTY, CITY, SCHOOL, DRAINAGE, PARK AND OTHER CORPORATION TAXES DUE FOR THE YEAR 1924, ON PERSONAL PROPERTY SITUATED IN THE TOWN INDICATED BELOW TO-WIT:

Name and Address	Assessed in Town of	Dist.	Estimate or Return	Volume Item No.
E. A. Jones, 43 University Av., Chicago, Ill.	Hyde Park		(R)	41-1841

MAKE ALL CHECKS PAYABLE TO
P. J. CARR, COUNTY COLLECTOR.
PERSONAL PROPERTY TAXES FOR 1924
ORIGINAL
DO NOT DETACH

AMOUNT RECEIVED

ASSESSED VALUATION	TOTAL TAX NOW DUE
153	12 8

REMITTANCE BY MAIL MUST BE BY DRAFT ON CHICAGO, MONEY ORDER OR CERTIFIED CHECK.

RETURN THIS BILL WITH YOUR REMITTANCE. DO NOT DETACH STUB.

PAID

The value of real estate is fixed by the assessor. The *assessed* value is usually lower than the *real* value. Land and buildings owned by churches, schools, and government are not taxable. By comparing the amount of money the government needs with the total assessed value of property, the assessor computes a rate per cent (tax rate). The individual taxes are then determined from this rate.

Exercises

1. A man owns real estate assessed at \$9500. The tax rate was \$.00643 on each dollar. Find the tax.

Solution: Tax on \$9500 = (\$9500 × .00643) = \$61.09.

2. At the rate given in Exercise 1, find the taxes on the following assessed values: \$5350; \$16,480; \$35,400.

3. Find the amount of the taxes on a piece of real estate assessed at \$1550 at \$2.75 on each \$100 of assessed valuation.

Suggestion: Divide \$1550 by 100 and multiply the result by \$2.75.

4. Find the taxes on the following assessed values at the rate given in Exercise 3: \$6850; \$15,450; \$11,360.

5. In a certain city the tax rate is \$22.42 on each \$1000 of assessed valuation. Find the tax of a man whose property is assessed at \$13,000.

6. Find the taxes of a man whose real estate is valued at \$9825 and whose personal property is \$485, the tax rate being 1.5 per cent.

7. If the tax rate in a city is 1.4 per cent of the assessed value, find the tax on property valued at \$18,450.

8. Compute the taxes on a piece of real estate assessed at \$8925 at \$8.25 per \$100 on one-half of the assessed value.

9. Find out what method of determining the value of property is used in your city and how the tax rate is computed.

10. Find out for what purposes the taxes are spent by your government. The adjoining table shows how the taxes are divided in one of our great cities. Tell what per cent of the tax is given to each item.

Interest on debt	\$ 9.50
Library	1.25
Pensions	2.00
Sanitarium	1.50
Parks	10.00
School buildings	10.00
Education	19.00
Sanitary district	5.50
County	9.75
State	14.25
City	17.25

11. Show that a tax rate of \$1.30 per \$100 is the same as 1.3 per cent of the value.

12. Find the tax rate used in your city and compute the taxes of people whose real estate valuations are as follows: \$13,420, \$9460, \$22,510, \$32,850.

13. A village is to be annexed to an adjoining city. State some of the advantages the people of the village are going to gain. Should their rate of taxation be increased?

14. In a small city the total assessed valuation of all property is \$2,321,600. It is necessary to raise a tax of \$38,074. Find the tax rate.

Suggestion: Divide the amount to be raised by the assessed value.

15. A man's tax was \$122.57 and his property was valued at \$8072. What should be his neighbor's tax who owns property valued at \$12,450?

16. The assessed value of property in a city is \$5,160,000 and a tax of \$64,500 is to be raised. Find the tax rate.

17. Sometimes tax rates are expressed as so many mills on each \$1 of assessed valuation. Thus we have the following ways of expressing tax rates:

A tax rate of \$16.40 on a \$1000,
which is equal to
a tax rate of \$1.64 on a \$100,
which is equal to
a tax rate of 1.64 cents on a \$1,
which is equal to
a tax rate of 16.4 mills on a \$1, since 1 cent is equal to 10 mills.

Express a tax rate of \$1.86 on \$100 of assessed valuation in each of the ways given above.

130. Special assessments. Some improvements, such as laying cement walks, paving alleys and streets, and putting in sewers, are of direct benefit to the ad-

joining property. The owners must then pay a special tax to help pay for those improvements. Such a tax is called a *special assessment.*

Exercises

1. A man purchased a vacant lot in an undeveloped neighborhood. Two years later the street was paved, for which the property owners were charged $4.25 a front foot. A sewer was built costing $2.40 a front foot, and sidewalks were constructed for which each was charged $1.35 a front foot. If the lot is 50 feet wide and if its original cost was $1100, what is the total cost of the lot not counting real estate taxes and interest, after the improvements were put in?

2. Why do the owners of the adjoining property have to pay the cost of paving the street?

3. Why do the owners of the adjoining property not have to pay the cost of a library or park?

131. How the national government collects money. There are several ways in which the national government collects money:

1. One of the sources of revenue is the *income tax.* As the name suggests, it is collected on net incomes above a certain amount. Those who have large incomes pay more than those whose incomes are small. In computing the income tax certain deductions are allowed: $1,000 to a single person; $2500 to a head of a family whose net income is not more than $5000; $2000 to a head of a family whose income is $5000 or more; $400 for each person in the family under 18 years of age or incapable of self-support. In the year 1924 the government also allowed a reduction on the taxes which was equal to 25 per cent of the income actually earned.

2. Another way for the government to raise a considerable amount of money is to charge a tax on imported articles. This is known as *import duty*, tariff, or custom. Goods that are purchased in other countries and brought into the United States are subject to this tax. The rates are fixed by the tariff law. Besides deriving an income from the tax on imports, the government uses it to protect certain industries. It may be possible to buy in England for $26.50 a suit of clothes costing $35.75 in the United States. A tax of $9.25 on suits of that type enables the clothing industry in this country to compete with that of England.

The rate of tax may be determined according to the value of goods (*ad valorem duty*). This is done by taking a rate per cent of the purchase price of the goods. Or the rate may be fixed regardless of value, a certain amount per pound, or per piece. The tax is paid to the *customs collector* when the goods are brought into this country.

3. A national revenue is obtained from taxes on certain articles made in this country, as cigars, medicines, playing cards, and drugs.

Exercises

1. Find the duty on a fur coat valued at $285 if the rate is 25 per cent *ad valorem*.

2. How much duty was paid on 205 bushels of barley bought in Canada, if the rate is 20 cents per bushel?

3. What is the duty on a brussels rug if the original cost was $45 and the duty 55 per cent *ad valorem?*

4. Find the duty on each of the following:

Article	*Purchase Price*	*Rate*
A watch	$41.25	45% *ad valorem*
1 dozen pairs of hose	5.80	35% *ad valorem*
Sugar, 500 pounds	30.00	2.2c per pound
60 yards of lace	25.20	90% *ad valorem*
16 dozen of ladies' gloves		$4 per dozen
500 pounds of butter		8c per pound

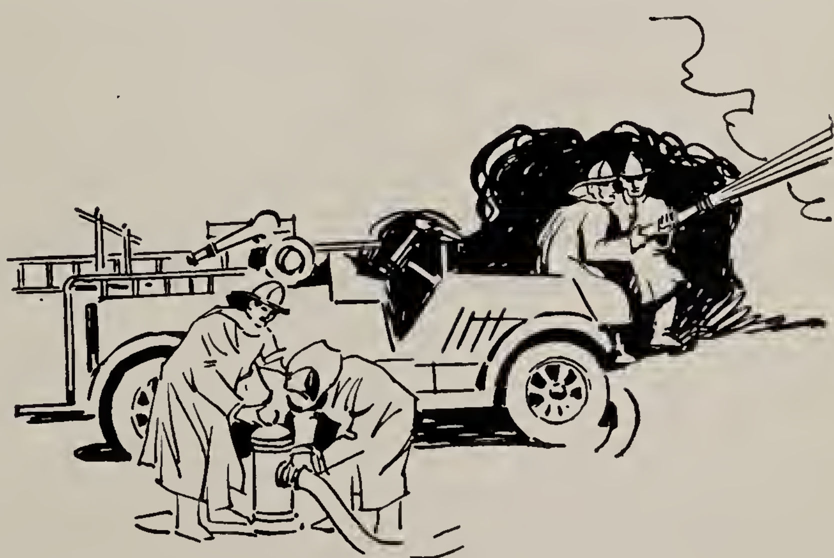
5. Who is benefited when potatoes are imported without being taxed? Who loses by it?

6. If the duty on automobiles is 25 per cent *ad valorem*, what will be the duty on a French automobile costing $2300?

7. If the duty on horses is 20 per cent *ad valorem*, what will be the cost of six horses purchased in Canada for $150 each?

Insurance

132. How property owners are protected against loss by fire. Mr. Olson's house was destroyed by fire, and he lost overnight what has taken him many years to save. It may cause him to suffer hardships and even ruin him financially. If all the property



owners of his city would join and pay back to him his loss, it would cost them a very small sum of money.

For his $7000 building each of the 35,000 property owners would have to pay only 20 cents. The property owners could then make an agreement to reimburse anyone whose property should be destroyed or damaged by fire in the future. It would be necessary to hire somebody to estimate the value of the property and the amount of damage done, to collect the money, to turn it over to the proper person, and to make out a full statement of all the money transactions.

To relieve the property owners of this responsibility companies have been formed that make it a business to guarantee protection against loss by fire. They are called *insurance companies.* They give the owner a contract, called an *insurance policy,* in which the company agrees to pay for losses from fire a sum of money equal to an amount agreed upon. For this protection the insured pays a fixed sum of money, called the *premium.* Since the total of the premiums must be more than the sum of all losses, the premiums are made sufficiently high to enable the company to pay all expenses in conducting the insurance business, to pay all losses, and to make a profit. Thus, through the insurance company the many who have no loss through fire pay the losses that actually occur.

The rate of insurance is determined by experts. It depends on the quality of the building, the purpose for which it is used, its location, the kind of water system and fire department, and other conditions. The rate of insurance for a building in a city with a good fire department is lower than the rate for the same kind of building in the country. The rate for a brick building is lower than that for a frame dwelling.

Fire insurance not only protects the owner against loss, but adds to the value of the property. For example, it enables the owner to borrow money by offering the property as security, since the risk of loss through fire has been removed.

Exercises

1. Why is it that an insurance company can pay $6000 to the owner whose house is destroyed by fire, when he pays only a yearly premium of $25.00?

2. A house valued at $10,000 and insured for $8000 was damaged by fire to the extent of $3000. How much will the insurance company pay the insured? Give reason for your answer.

3. Is it necessary to insure a house for its full value?

4. For which of each of the following pairs of property would the rate be lowest under otherwise equal conditions: frame house, brick house; frame apartment, brick apartment; frame garage, brick garage? In each case give your opinion as to reasons for the lower rate.

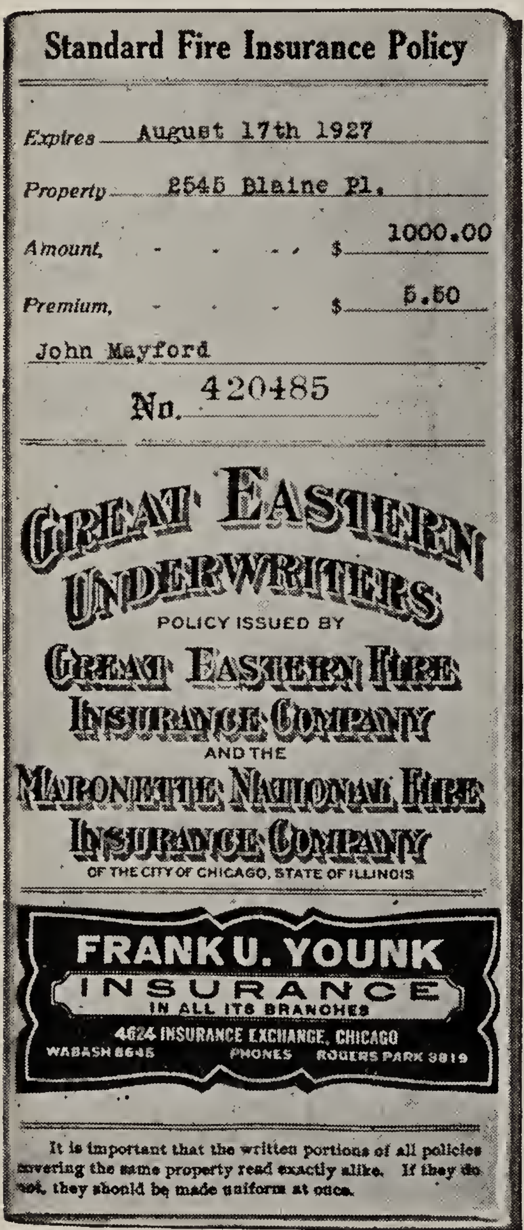
5. Name all the different factors which will be considered in determining the rate of insurance on a building.

133. Other kinds of insurance. Property losses may be incurred in ways other than through fire. A tornado sometimes destroys a whole section of a city. Hail, frost, floods, or insects may ruin our crops. People can get insurance against losses from any of these causes.

A copy of a standard fire insurance policy for insuring city and village dwelling property against losses by fire is shown on page 202. Try to secure a copy of such a policy and study it.

Exercises

1. Mr. Johnson insures his property for \$8000 for 3 years. The insurance rate is \$.16 per \$100 for 1 year. For 3 years it is $2\frac{1}{2}$ times as great as for 1. Find the premium.


Standard Fire Insurance Policy

Expires August 17th 1927

Property 2545 Blaine Pl.

Amount, \$ 1000.00

Premium, \$ 5.50

John Mayford

No. 420485

GREAT EASTERN UNDERWRITERS

POLICY ISSUED BY

GREAT EASTERN FIRE INSURANCE COMPANY

AND THE

MARQUETTE NATIONAL FIRE INSURANCE COMPANY

OF THE CITY OF CHICAGO, STATE OF ILLINOIS

FRANK U. YOUNK

INSURANCE

IN ALL ITS BRANCHES

4624 INSURANCE EXCHANGE, CHICAGO

WABASH 8645 PHONES ROGERS PARK 3819

It is important that the written portions of all policies covering the same property read exactly alike. If they do not, they should be made uniform at once.

2. Fred's father insures his frame garage for \$460 for 5 years. The insurance rate is \$3.00 per \$100 for 1 year. For 5 years it is 4 times as much as for 1 year. Find the premium.

3. The rate of insurance on a \$12,000 brick flat is \$.24 per \$100. Find the premium for 3 years, if the amount for 3 years is $2\frac{1}{2}$ times as great as for 1 year.

4. A \$6000 house is insured for 80 per cent of its value. The rate of insurance is \$.50 for each \$100. How much is the premium?

5. Mr. Black's house is worth \$8000. It is insured for 75 per cent of its value at a rate of \$.50 per \$100. What is the premium?

6. If the rate for 3 years is $2\frac{1}{2}$ as much as for 1 year, and if the rate for 5 years is 4 times as much as for 1, complete the following table of rates per \$100:

Type of Structure	*1 Year*	*3 Years*	*5 Years*
House, brick	\$0.16		
Frame	.50		
Apartment, brick	.24		
Frame	.75		
Garage, brick	.28		
Frame	.78		

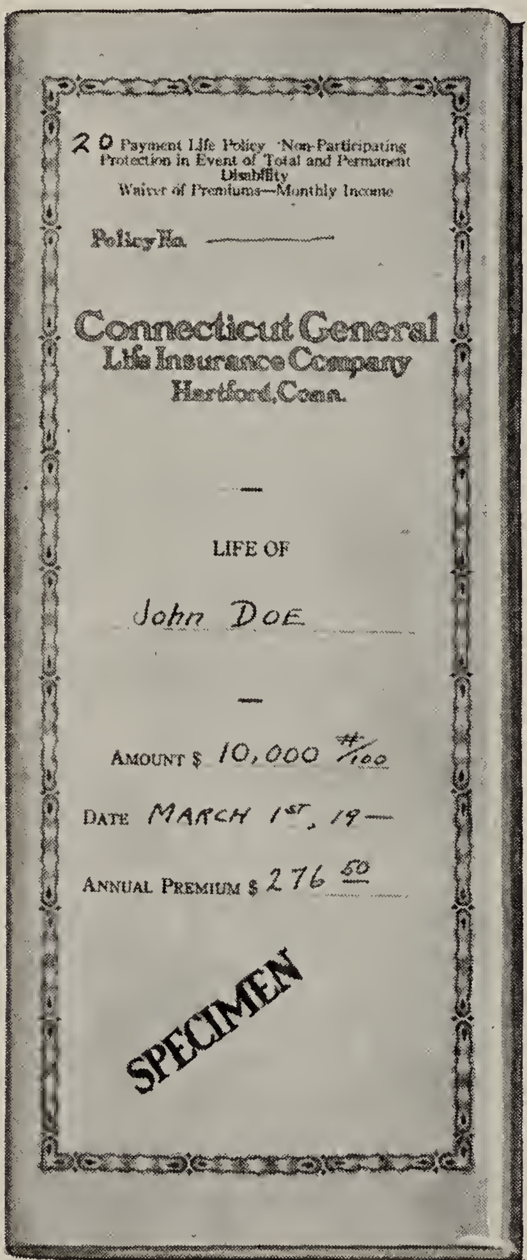
7. Why does the insurance company hesitate to insure property for its full value, thus making the owner bear a portion of the risk?

134. How life insurance protects the family. William's father lost his life in an automobile accident. The family was left without income, as he was the only one earning money. Fortunately, several years ago he had made provision for the support of his wife and children after his death, and they received a check for $12,000 from a life insurance company.

Any person whose family depends on him for support should carry life insurance in an insurance company. By making contracts with a large number of people who pay a small sum of money each year, the company is able to pay a large sum to each of the families of those individuals who die during the year. Some of the companies insure as many as a half million people. Statistics show fairly well how many individuals out of 500,000 ordinarily die during one year. From these facts the company can determine the rate which must be charged for insurance to enable them with certainty to pay all claims, pay expenses, and have a profit left over. Life insurance in a good insurance company is, therefore, a safe investment.

135. Kinds of life insurance. The simplest of the different kinds of policies issued by life insurance companies is the *ordinary life policy.* The insured agrees to pay a fixed premium annually during lifetime, and the company to pay a definite sum upon the death of the insured to the person designated in the policy.

A *limited payment life policy* is one in which the insured stops paying premiums after a limited time, *e.g.*, 10 or 20 years. The premium is, of course, larger than that of a straight life policy because the payments on a limited payment life policy are all made in a few years of the life of the insured. Many would rather pay a little more each year for a few years than keep on paying a smaller premium into old age.



Finally, there is a policy for those who like to have insurance protection for the family and at the same time derive some personal benefit from it. This policy is known as *endowment policy*. The insured pays a premium yearly for 10, 15, or 20 years. The company agrees to pay a certain sum of money at the time of his death or at the expiration of the period agreed upon. The good feature of this policy is that it induces people to save some money each year while they might otherwise spend it thoughtlessly.

The following table is taken from the rate book of a good insurance company. It gives the rates for $1000 for three kinds of life insurance:

TABLE OF ANNUAL PREMIUMS PER $1000 OF POLICY

Age	Ordinary Life	20-Payment Life	20-Year Endowment
20	$13.48	$31.22	$40.85
21	13.77	31.44	40.89
22	14.08	31.68	40.94
23	14.41	31.97	40.99
24	14.75	32.25	41.04
25	15.10	32.54	41.10
26	15.48	32.90	41.16
27	15.88	33.27	41.23
28	16.29	33.63	41.31
29	16.73	34.07	41.39
30	17.19	34.52	41.49
31	17.72	35.04	41.63
32	18.26	35.60	41.79
33	18.84	36.20	41.94
34	19.45	36.87	42.13
35	20.11	37.58	42.33
36	20.84	38.36	42.54
37	21.60	39.24	42.78
38	22.42	40.14	43.04
39	23.29	41.14	34.32
40	24.21	42.18	43.65

136. Cash value of a policy. If the insured is unable to keep his contract and to pay the premiums on his policy, he does not have to lose all of the money paid in before that time. The company pays to him at the surrender of the policy a fixed sum, known as the *cash surrender value.* The following table gives the rates, per $1000 of insurance, and the respective cash surrender values of the policies:

SCHEDULE OF ORDINARY LIFE INSURANCE RATES AND CASH VALUES PER $1000

Age	Annual Rate	Cash Value		
		Three years	*Five years*	*Ten years*
15	$15.97	$10	$22	$54
16	16.26	11	23	56
17	16.54	11	24	59
18	16.86	12	25	61
19	17.19	12	26	64
20	17.52	13	27	67
21	17.88	14	28	70
22	18.25	14	30	73
23	18.65	15	31	76
24	19.06	16	32	79
25	19.48	16	34	82
26	19.92	17	35	86
27	20.41	18	37	90
28	20.90	18	38	93
29	21.43	19	40	97
30	21.97	20	42	102
31	22.56	21	43	106
32	23.17	22	45	111
33	23.81	23	47	115
34	24.50	24	49	120
35	25.22	25	52	125
36	25.99	26	54	131
37	26.79	27	56	136
38	27.65	29	59	142
39	28.57	30	61	148
40	29.53	31	64	154

137. Dividends. The premium paid by the insured is large enough to enable the company to meet all pay-

ments and to have a profit left over. Some policies entitle the insured to share in the profits. They are called *participating* policies. Each year a sum of money is returned to the insured, thereby reducing the premium which he has to pay. These refunds are called dividends.

Exercises

1. Find out what you can about the following kinds of insurance:

Sickness and accident insurance, tornado insurance, employers' liability insurance, unemployment insurance, fidelity and surety insurance.

2. By means of the table (page 205) tell the annual premium for a 20-year payment life policy to be paid by a man 30 years old; 25 years; 32 years.

3. What kind of insurance gives the greatest protection for a given premium?

4. Find the yearly premium on an ordinary life insurance policy of $5000 for a man 32 years of age; 26 years; 30 years.

5. A man carried a 20-payment life participating policy for $3000 which he took out at the age of 27. The premium was $149.02. He received 19 dividends averaging $26.20 per year. How much did he actually pay for his insurance?

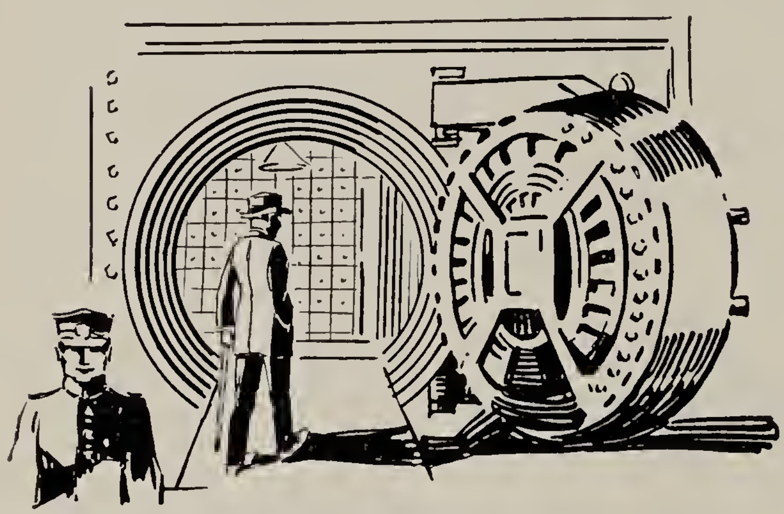
6. A man 32 years old takes out $5000 of 20-year endowment insurance. Find the difference between the amount of the policy and what he will pay in during the 20 years.

7. A man took out a 20-year endowment policy at the age of 26. The company declared dividends averaging 16 per cent of the premium. Find the difference between what he paid in premiums and what he received from the company.

8. Find the cost of the premium of an insurance policy for a farm house, barn, and several other buildings valued at $12,000 if insured for $\frac{2}{3}$ of the actual value and at a rate of 75 cents per $100.

Banks and Banking

138. Why banks are needed. People who save money find it not safe to carry the money around with them or to hide it away. There is too much danger of being robbed. One of the several things banks do is to accept money and to keep it in a safe place. This service to people who have money relieves them of a great deal of anxiety and concern. Banks also rent



safety deposit boxes in which valuables, such as securities, important papers, or jewelry, may be kept for safe keeping.

The money intrusted to them, together with capital of their own, makes it possible for banks to loan money to people and to pay all business expenses with the profits derived from the payment of interest on loans.

Exercises

1. Name some of the banks in your city.

2. The business manager of the school annual takes in several hundred dollars the day the *Annual* is sold to the pupils. He may lock the money in his desk, or take it home with him, or take it to a bank. Which should he do? Give your reasons.

3. Find out how banks are protected against robbers or loss by fire.

139. Different kinds of banks. To protect people against losses when they put money into banks, the government supervises the banks carefully. In organizing and conducting their business bankers must obey laws made especially for that purpose.

The name indicates the character of a bank. *National banks* are under the supervision of the national government and are inspected by national officers. *State banks* are organized under the laws of the state and examined by state officers. There are a few central banks in the national banking system, called *federal reserve banks.* The government requires a very large capital before such a bank can start to do business. Federal banks are authorized to issue paper money. *Private banks* are not under government inspection. They are owned by individuals or companies. *Trust companies* generally do a banking business. As the name suggests, they furnish services that are in the nature of a trust, as settling estates and acting as guardians of minors.

Two of the most important departments of a bank are the savings department and the commercial department. We shall learn more about them.

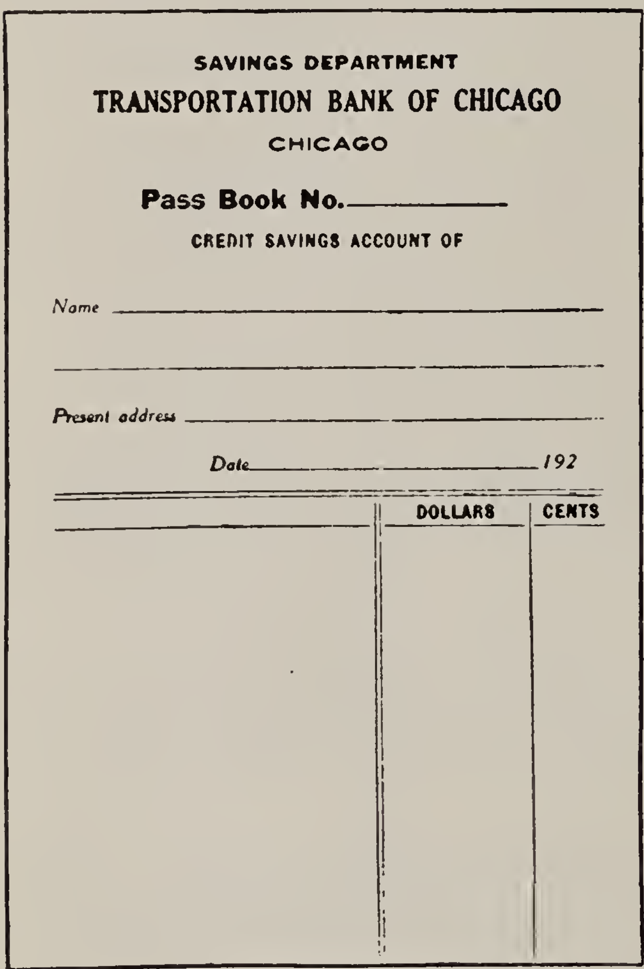
Exercises

1. Examine some paper money to see if you can find bills issued by a bank.

2. Find out how many federal reserve banks there are and where they are located.

3. Why do people put money into private banks when there are other banks in the city?

140. How the business of a bank is conducted. We have seen that banks receive money for safe keeping and lend money, and that some issue money. The responsibility for conducting the affairs of the bank rests with the president, the cashier, and the receiving and paying tellers. They, in turn, work under a board of directors.



SAVINGS DEPARTMENT
TRANSPORTATION BANK OF CHICAGO
CHICAGO

Pass Book No.________

CREDIT SAVINGS ACCOUNT OF

Name ____________________

____

Present address ____________________

Date____________192

	DOLLARS	CENTS

141. Opening a bank account. When opening an account with a bank the new customer is asked to fill out a card giving his signature, address, references, or other matters of details. This card is filed for the purpose of later identification. The customer then receives a bank book on which the teller of the bank enters the various amounts (deposits) given to him from time to time by the customer (depositor). The bank book is the depositor's receipt for the money deposited, and is kept by the depositor. To make the deposit a special slip has to be filled out. This, together with the money and bank book, is given to the teller who enters the date and amount on the page of the book, which is then returned to the customer.

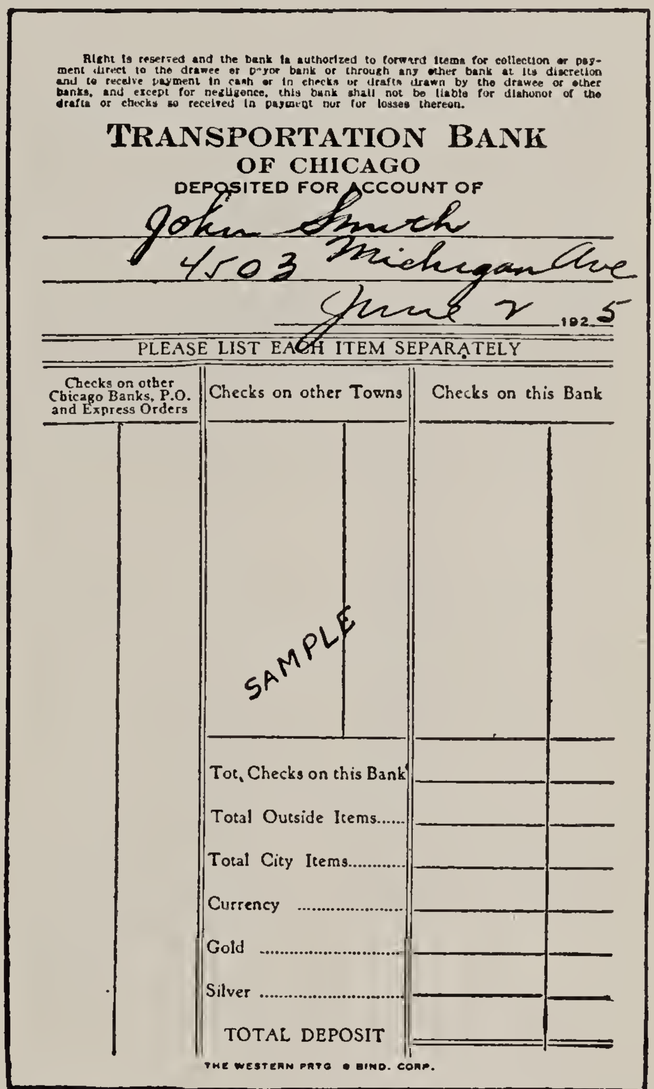
There are two kinds of accounts, the checking account and the savings account.

Exercises

Write the following deposit slips for H. Jones:

1. Monday, June 6, 1925: Checks \$18.60, \$10.50, \$9.75, \$15.00; currency \$85; gold \$15; silver \$6.70.

2. Tuesday, June 7, 1925: Checks \$5.40, \$22.67, \$13.25, \$12.50; currency \$114; gold \$25; silver \$18.60.


Right is reserved and the bank is authorized to forward items for collection or payment direct to the drawee or payor bank or through any other bank at its discretion and to receive payment in cash or in checks or drafts drawn by the drawee or other banks, and except for negligence, this bank shall not be liable for dishonor of the drafts or checks so received in payment nor for losses thereon.

TRANSPORTATION BANK
OF CHICAGO
DEPOSITED FOR ACCOUNT OF
John Smith
4503 Michigan Ave
June 7 1925

PLEASE LIST EACH ITEM SEPARATELY

Checks on other Chicago Banks, P.O. and Express Orders | Checks on other Towns | Checks on this Bank

SAMPLE

Tot. Checks on this Bank
Total Outside Items
Total City Items
Currency
Gold
Silver
TOTAL DEPOSIT

THE WESTERN PRTG & BIND. CORP.

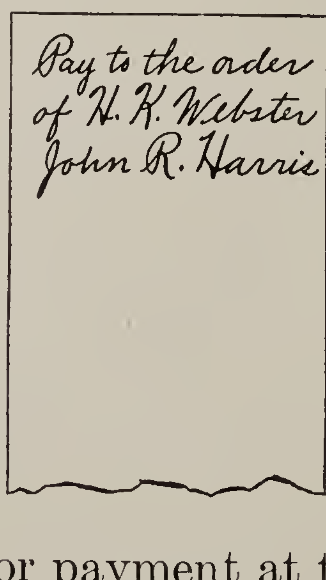
142. Checking accounts. If a depositor wishes to keep his money in a safe place, so that he may draw from it at any time, he opens a *checking account.* The teller enters the various deposits on the bank book, and the bank keeps a record of all deposits and withdrawals so as to know each day how much money the depositor has. For this service the bank has the use of the money, usually without paying interest, and frequently requiring that the depositor keep a fixed balance in the bank.

Money is withdrawn by *checks.* They come in the form of check books. In some check books each page

contains a check blank and a stub. After filling out both, one tears the check off while the stub remains as a memorandum of the check. The stubs show the money deposited and withdrawn and the purposes for which money has been spent. The following is a sample stub and check.


No
Date
To
For
DOLLARS CENTS
Bal brought forwd.
Amount deposited
Total
Amount this check
Bal carried forwd.
CHECKS IN ANY WAY ALTERED OR CHANGED WILL NOT BE HONORED
TRANSPORTATION BANK OF CHICAGO 2-70
CHICAGO, ILL., 192 No.
PAY TO THE ORDER OF $
DOLLARS

The check may be made payable to any person to whom a payment is to be made. He may deposit it at his bank like money, or cash it at any bank, provided he identifies himself and *endorses* the check by writing his name across the left end of the back of it, as shown at the left.



Checks may be made payable to *self* or *cash*. In the last case no endorsement is necessary if the signer himself presents the check for payment at the bank.

A person receiving a check may give it to another, making it payable to him.

Exercises

1. Fill out a bank deposit slip; a withdrawal receipt.

2. Make out a check and fill in the blanks of the stub. Find out what precautions are used in filling out a check blank to prevent another person from changing the amount.

3. Find out the minimum balance your neighborhood bank requires on checking accounts.

4. Endorse a check made out to you. Why should the endorsement be always at the left end of the check?

5. Why should a check not be endorsed until it is time to deposit it?

6. Why is it not necessary to endorse a check payable to "cash" or to "bearer?"

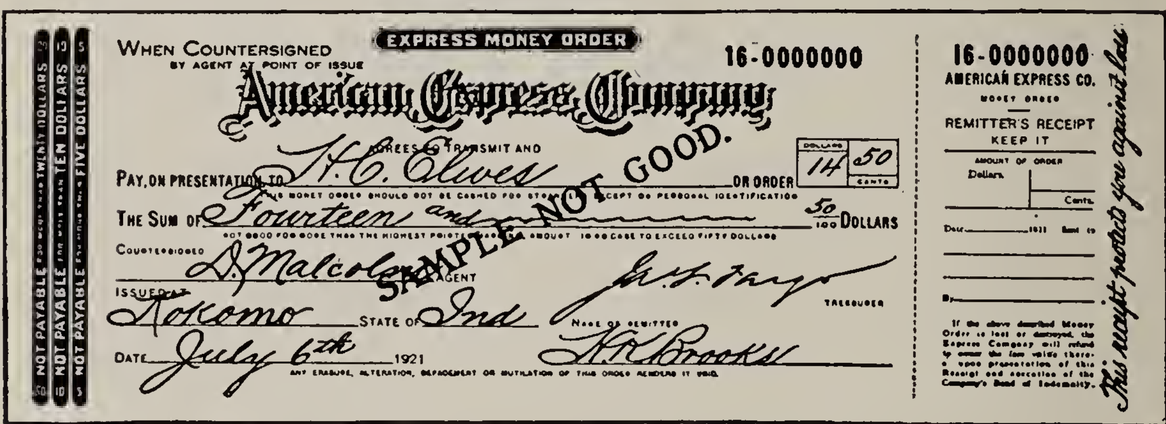
7. Why is the sum of money on a check written both in figures and in words?

8. Why is it best to deposit or cash a check as soon as possible?

143. Paying bills at a distance. The use of checks makes it convenient for people to pay their bills by sending checks through the mail. The checks are deposited or cashed in the various banks in the city. It then becomes necessary for each bank to return to the others the checks drawn upon them. Since this is not a practical thing to do, all the banks of the city form a *clearing house association.* Every day the checks received are sent to the clearing house. Each bank is given credit for the total amount of the checks sent in and at the same time charged with the checks made out upon them.

At the beginning of the month all the paid checks of the previous month with a complete statement of deposits and withdrawals are returned by the banks to the persons who wrote them. These paid checks are preserved because they show that the bills have been paid and may therefore be used as receipts.

An arrangement similar to the clearing house exists between banks for checks sent from one city to another. Merchants sometimes do not like to accept personal checks coming from a great distance. In that case bills may be paid by sending money orders issued by the post office or by express companies.



Exercises

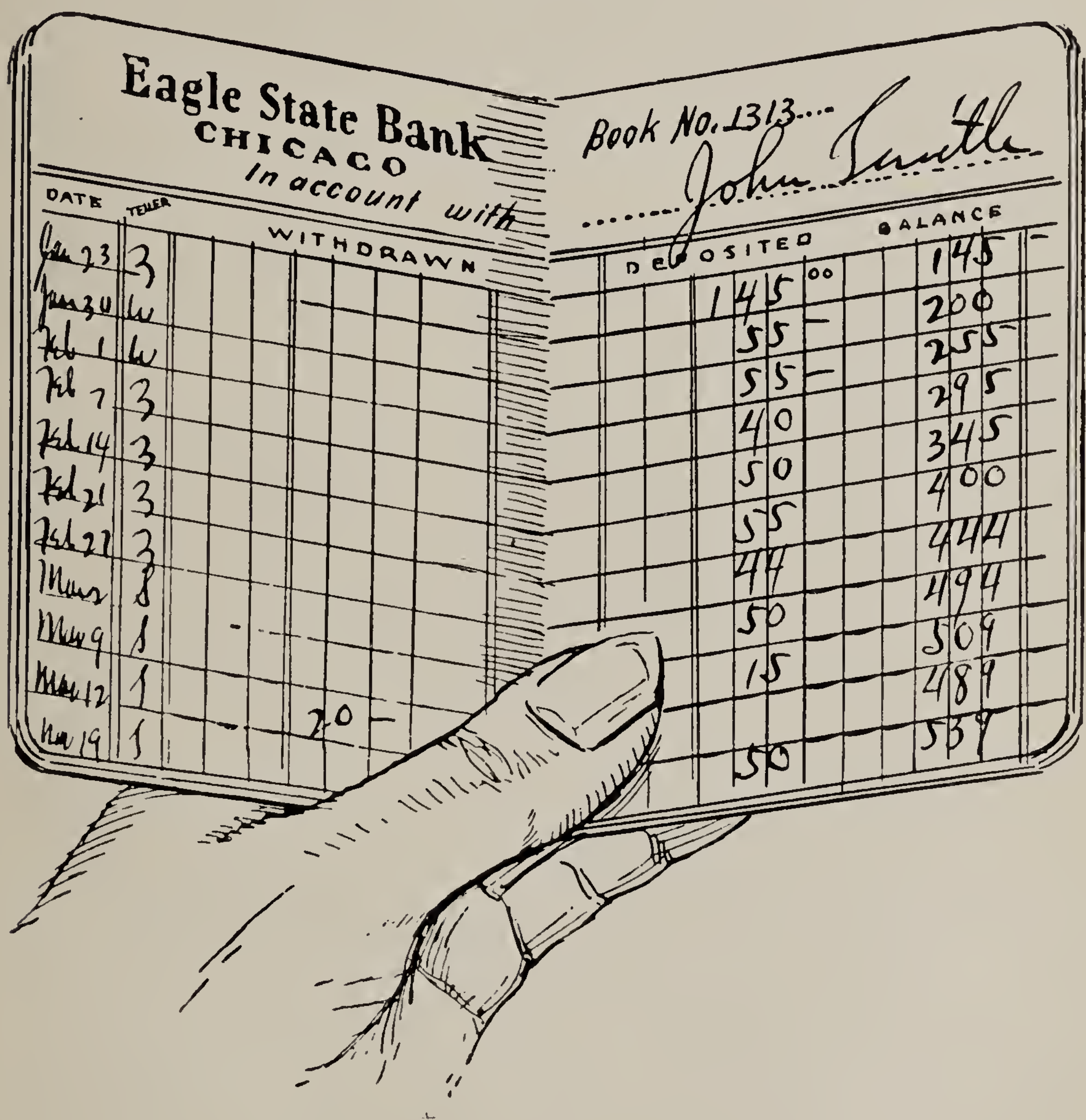
1. Why is a canceled check accepted in place of a receipt?

2. To whom is the express money order above payable?

3. Why are bills at a distance not always paid by checks, but by money orders?

144. Savings accounts. Amounts of money may be placed on interest by opening a savings account. The savings bank makes it possible to keep sums of money in a safe place and at the same time earn a little money

with it. The rate of interest is usually low, about 3 per cent or 4 per cent. Money can be withdrawn only if the pass book is presented.



Some people put part of their earnings into the savings bank until they have an amount large enough to invest where it will bring greater returns. Interest on savings deposits is paid according to rules, which should be studied carefully by those who open a savings account. The following rules are copied from a bank book:

"Twice each year interest is paid on all savings deposits then in the bank, which have remained on deposit for one month or more.

"No interest is paid for parts of a month, or part of a dollar, or for any part of the half year included between the first days of January and July, or July and January, as the case may be, on sums withdrawn between these periods.

"Interest is not paid on average balances; and all withdrawals between the interest days are deducted from the first deposits.

"On deposits made during the first ten days of January and July, and during the first five days of every other month, interest is allowed from the first day of the month in which the deposit is made. On deposits made after the above-named days of each month interest is allowed from the first day of the month following the deposit."

It should be noted that the interest rates of savings banks are low, and that the interest will be reduced considerably if the rules above are not clearly understood. There are good reasons why a bank cannot pay more than 3 per cent or at most 4 per cent interest. To make a profit, the bank reinvests the deposits in high-grade securities usually not paying more than 5 per cent or 6 per cent. Banks also loan money to individuals or to business houses charging them as much as 7 per cent interest. However, not all of the deposits can be invested, as a large part must be kept on hand to conduct the business of the bank.

Exercises

1. Compute the interest at 3 per cent on the following savings account, observing the rules stated above:

Date	*Teller*	*With-drawals*	*Deposits*	*Balance*
July 8	*B*	. . . .	$ 60.00	$ 60.00
Aug. 26.	*CM*	. . . .	42.00	102.00
Sept. 4	*MJ*	. . . .	25.75	127.75
Oct. 1	*CM*	. . . .	48.60	176.35
Nov. 17.	*B*	. . . .	50.00	226.35
Nov. 28.	*B*	$ 72.00	. . . .	154.35
Dec. 3	*CM*	. . . .	45.00	199.35
Interest, Jan. 1	. . . . . .	. . . .	. . . .	. . . .

2. Make an account similar to that of Exercise 1, using the following deposits and withdrawals and compute the interest at 3 per cent. On January 1, Mr. Carr deposited $28, and on January 18 he withdrew $5. On each of the following months he deposited respectively $13, $8, $22, $19.50, $25. He withdrew $10 on July 1.

3. Starting with a balance of $83 in a savings bank Elizabeth's mother made the following deposits:

Jan. 12	$ 9.00	April 5	$13.00
Feb. 5	15.00	May 1	29.00
March 4	8.00	June 16	10.00

Compute the interest coming to her on July 1.

Investments

145. Using money to earn more money. We have seen that a savings bank offers a safe place for keeping sums of money and pays some interest to the depositor.

As soon as one has accumulated a considerable amount of money the way is open for finding an investment where it will earn more than 3 per cent or 4 per cent interest. It now becomes necessary to make a study of ways of investing money without running too great a risk of suffering losses. Before making an investment it is always well to get the advice of a reliable banker or broker. For many people are defrauded each year by loaning money to enterprises which fail. One should be especially careful about all investments that offer too high a rate of interest. They are usually uncertain and speculative, while investments paying a fair rate of interest as a rule involve a small risk.

146. Investing money in a mortgage. When people who own property are in need of money they can borrow money by offering the property as security. The investor who lends the money receives a *mortgage* as security. Mortgages usually yield interest at the rate of 6 per cent. Good mortgages are considered a safe investment. The loan must be paid back on time or the lender can have the property sold and retain his money out of the money received from the sale. When buying a mortgage one should take several precautions. The loan should not exceed one-half of the value of the property offered as security. All taxes on the property must be paid. The insurance on the buildings must be made out in favor of the holder of the mortgage. There should be no other mortgage on the property. The mortgage should be officially recorded in the bureau of records.

Exercises

1. What is the semi-annual income derived from a $1300 mortgage on property worth $8000 at the rate of 6 per cent?

2. A man wishes to borrow $4000, offering as security a piece of property valued at $7000. He will give the lender a mortgage paying 7 per cent interest. Is this a safe investment? What per cent of the value of the property is the mortgage?

3. Why is it necessary before buying a mortgage that the insurance be in the buyer's favor, and that all taxes be paid for?

4. Why is a good mortgage considered a better investment than a savings account?

5. Find the yearly income on the following: a mortgage for $1300 at 6 per cent; one for $2500 at 5 per cent; one for $1800 at $5\frac{1}{2}$ per cent.

147. Corporations. When a number of men wish to go into business as a group, they may organize a *corporation*. They apply to the secretary of state for a *charter* which defines the conditions according to which the corporation is to carry on business. Many large business enterprises, such as running railroads, manufacturing, and mining, are managed by corporations.

148. Bonds. The charter permits a corporation to borrow money if capital is needed. The required sum is divided into $500 or $1000 amounts, known as *bonds*. Bonds are really promissory notes with the property of the corporation as security, promising to pay a fixed sum of money at a given time with interest at a given rate. Interest has to be paid to holders of bonds before the profits of the business are distributed. Good bonds are therefore considered a safe investment. If the corporation should become bankrupt, bond holders may

demand the sale of the corporation's property and be reimbursed from the proceeds. The market value of bonds varies comparatively little. Interest on bonds is always paid promptly.

The following advertisement of the bond department of a bank calls attention to matters of interest to the prospective investor:

BUY THE BONDS YOUR BANKER BUYS

The bond described below is issued by a company long established in a sound, basic industry vital to the country's business and domestic needs.

In our opinion, the issue constitutes a high-grade investment.

Bonds are available in denominations of \$1000 and \$500.

Prices, to which interest should be added, are subject to change.

PENNSYLVANIA COLLIERIES COMPANY

First Mortgage 6s

Price $98\frac{1}{2}$, to yield 6.12%

These bonds, of which \$5,000,000 are outstanding, are secured by a first mortgage on properties valued at over \$17,000,000 and estimated to contain over thirty-seven million tons of recoverable coal.

Situated in the heart of the rich Pennsylvania anthracite coal field, the company's holdings consist of six properties, the oldest of which has been in operation for over sixty-five years. The mines are served by the Philadelphia & Reading, Lehigh Valley, and Delaware & Hudson railroads.

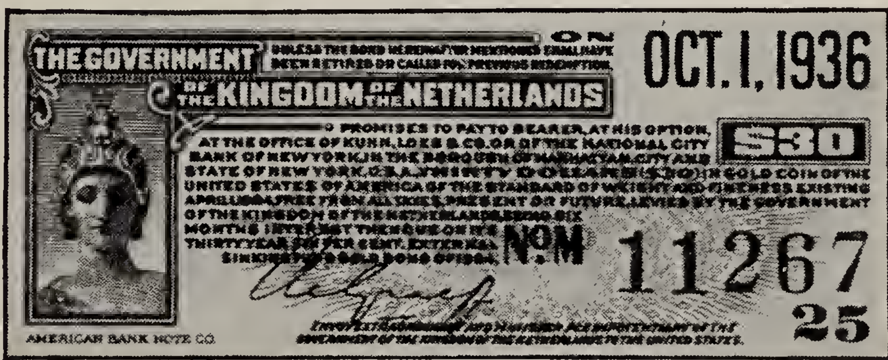
Throughout the East the laws prohibit the use of other than anthracite coal, so that the Pennsylvania Collieries Company serves an exclusive market.

A sinking fund per ton of coal mined will, it is estimated, retire the entire issue by maturity. After meeting the sinking fund requirements, the bond interest was earned nearly four times in 1924.

The bonds mature November 1, 1944. Interest is payable May 1st and November 1st of each year.

Interest does not depend on the market value of the bond. The bonds above are to yield 6.12 per cent interest on the par (face) value. If a $1000 bond is sold for $985, the interest of $61.20 is greater than 6.12 per cent of the investment. When payment of the bond is made in 1944, it will bring $1000, or $15 more than the original cost.

Coupon bonds have small slips (coupons) attached. There is a coupon for each interest date, calling for the interest accumulated since the preceding date. On the date given the coupon may be clipped and presented for payment or deposited in a bank on account.

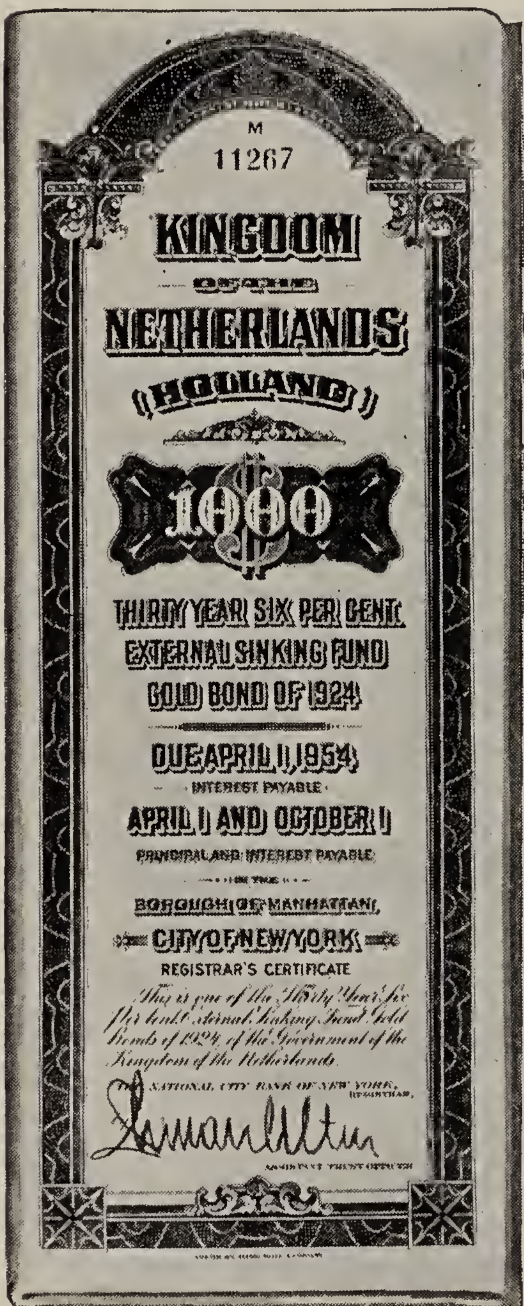


A *registered bond* contains the name of the purchaser. Before such a bond can be transferred to another person it must be endorsed by the owner and the name must be changed on the books of the company which issued it. The interest is sent directly to the owner.

Bonds are sold and bought by bankers and brokers not only for themselves but also for others. For this service they are entitled to a commission, which is usually one-eighth of one per cent of the par value of the bond. Many bankers and brokers buy back bonds sold by them, charging a small commission or they accept them as securities if the owner borrows money. This makes it possible to convert bonds into cash.

Exercises

1. Why should the investor before buying a bond consult a disinterested person, who is well informed about investments, *e.g.*, a banker or broker?



2. If a man buys one of the Pennsylvania Collieries Company bonds on dates other than May 1 or November 1, he must pay the interest earned since the preceding interest date. This will be returned to him on the following interest date. How much interest will he have to pay if the bond is bought on January 18?

3. Give several reasons why good bonds are considered a safe investment.

4. A man bought some $4\frac{1}{2}$ per cent bonds which yielded 5.3 per cent interest on the money invested. How much did he pay for a \$1000 bond?

5. What is the rate of income on a \$1000 seven per cent bond bought for 102.6?

6. Find the rate of income on the following \$1000 bonds:

6% bond bought at 87.5
$3\frac{1}{2}\%$ bond bought at 104.3
5% bond bought at par.

7. A man deposits \$15,000 with which his banker is to buy certain 4 per cent railroad bonds at 98.5, allowing a brokerage fee of $\frac{1}{8}$ per cent. How many bonds can be bought for the sum and how much cash will be left over?

8. When a bond is bought below par, is the rate of income greater or less than the rate of interest paid on the par value? What is the comparison when the bond is bought above par?

9. How many \$100 bonds sold at par and bearing 5 per cent interest secure an income of \$1400?

10. A Board of Education in a city sold \$30,000 worth of \$500 bonds paying 4 per cent interest. Find the annual interest paid on these bonds.

11. What is the cost of thirteen \$100 bonds quoted at \$98.10, if the brokerage is $\frac{1}{8}$ per cent and if they are bought 10 days after the interest date?

149. Stocks. The capital used in carrying on the business of a company is called *stock*. The stock is divided into a number of equal parts, called *shares*, and the owners of stock are *stockholders*. The owner of stock receives a *stock certificate* which states the number of shares held and the face value of each. He is really a part owner of the stock company and, therefore, entitled to his share of the profits. The distributed profits are called *dividends*. They are computed as a certain per cent of the par value of the stock. Two kinds of stock are issued, the *common stock* and the *preferred stock*. Dividends on the preferred stock, which are usually determined according to a fixed rate, are paid before any dividends on the common stock are paid. After dividends on the preferred stock have been paid the holders of the common stock share in the remainder of the earnings in proportion to their holdings. If a corporation is very successful, owners of common stock may receive more than owners of preferred stock. When a corporation is not making a

profit, no dividends are distributed. If it loses money, the stockholders may even be *assessed* a per cent of the par value of the stock to supply more funds.

When the business of a corporation is profitable, people may be willing to pay more than the par value for the stock. On the other hand, stocks may sell below par value when business is poor. Thus, prices of stocks

are very changeable, depending on dividends paid, business conditions, the security of the business and other factors.

An immense amount of business is carried on by stock companies. When a corporation expands its business, it may issue and sell additional stock to secure the necessary money. Stocks and bonds are therefore constantly bought and sold.

The *stock exchanges* in the large cities are places where stocks and bonds are bought and sold.

Exercises

1. What is the difference between stocks and bonds?

2. State the difference between common stock and preferred stock.

3. What are the advantages and disadvantages of common stock over preferred stock?

4. Why are people willing to pay more than par value for some stocks?

150. Some comparisons between stocks and bonds. Good government and municipal bonds are considered a safe investment because they are backed by the government or the property of the community.

The bonds of a corporation are safer than its stocks. For even a business enterprise which does not yield a profit or paying dividends on stock must pay interest on bonds.

If a company becomes bankrupt the bondholders may be reimbursed partly or entirely, while there may not be enough of a surplus to pay the stockholders. Furthermore, stockholders may be liable for the debts of the company to an amount equal to the par value of the stock they own.

The market value of bonds does not change greatly. If the owner has to sell, he can generally get a price nearly equal to what he paid. But since the values of stocks change, the owner cannot be sure as to the amount he may receive. A person who buys unsafe

stocks must keep in mind that he is speculating with his money.

The comparisons above show that an investor buying stock must be very careful, especially as to unsafe stocks promising very high returns. On the other hand, there are many corporations that pay regularly the dividends on their stocks, and some bonds that are not a safe investment.

Exercises

1. Find the annual income on 18 shares of stock paying 8 per cent dividends and purchased at $121.50 a share.

2. What is the annual income from 35 shares of National Bank stock, purchased at 120 and paying a quarterly dividend of 7 per cent?

3. A man bought some stocks at 135 which pay 8 per cent dividends, and invested the same amount of money in mortgages at 5 per cent interest. Find out which is the better investment.

4. The following table is a part of a report of the New York stock transactions given in the daily paper:

Description	*Sales*	*High*	*Low*	*Close*	*Net Chg.*
Adams Exp.	1,700	102	$99\frac{1}{2}$	100	-2
Am. Loco.	5,200	121	$119\frac{1}{8}$	$119\frac{1}{4}$	$-\frac{3}{8}$
Gen. Motors	37,100	78	$76\frac{7}{8}$	$77\frac{1}{4}$	$-\frac{3}{4}$
Mack Truck.	20,300	$148\frac{3}{8}$	$144\frac{1}{2}$	145	$-2\frac{7}{8}$
Pacific Gas.	1,200	$108\frac{1}{2}$	$106\frac{7}{8}$	$108\frac{1}{2}$	-2
Peoples Gas.	7,600	121	115	$118\frac{1}{2}$	$-3\frac{5}{8}$

The first column gives the number of shares sold, the second gives the highest price paid for stock, the third gives the lowest price, the fourth gives the price at the close of the market, and the last shows the increase or decrease in the price since the day before. Give the meaning of each line in the table above.

5. If the broker charges $12\frac{1}{2}$ cents a share for buying the stock, how many shares of General Motors stock could he have bought for you for \$578 as the market closed?

6. What is the rate of income on 8 per cent preferred stock purchased at 94?

7. A corporation has issued \$60,000 of common stock, \$60,000 of 6 per cent preferred stock, and \$30,000 of 5 per cent bonds. During the year the corporation has earned \$7000. Find the interest on bonds, dividends on preferred stock, and the per cent dividend that may be paid on the common stock.

8. Find the cost of 60 shares of stock selling at 122 if the broker charges $\frac{1}{8}$ of 1 per cent.

9. A man bought 100 shares of stock at 96 and sold them at 102, paying $\frac{1}{8}$ per cent brokerage. Find his profit.

10. If at the end of the year a railroad company declares a 10 per cent dividend on stock whose par value is \$100, how much does the owner of seven shares of stock receive?

CHAPTER X

EFFICIENT METHODS OF COMPUTATION

Short Methods of Multiplication

151. What you are going to study in this chapter. The purpose of Chapter X is to present various methods of computing that are valuable because they save time and effort. Many of these methods are employed by people who do a considerable amount of number work in practical business. You have previously seen the value of the abbreviated processes of multiplication and division. It will be to your advantage to study the devices taught in this chapter until you are able to make use of them with ease.

152. Multiplying by a power of 10. Examine the following examples and state how to multiply a number quickly by 10, 100, 1000, etc.

$647 \times 10 = 6470.$	$54.1 \times 100 = 5410.$
$3.7 \times 10 = 37.$	$.84 \times 100 = 84.$
$.036 \times 10 = .36.$	$12 \times 100 = 1200.$

The preceding illustrations show:

1. That a whole number is multiplied by 10, 100, etc., by annexing zeros;

2. That a decimal fraction is multiplied by 10, 100, etc., by moving the decimal point to the right.

Since in a whole number the decimal point is understood to be to the right of the unit digit, we may say

that in both cases *the number may be multiplied by 10, 100, 1000, etc., by moving the decimal one, two, three, or more places to the right, annexing zeros if necessary.*

Exercises

Multiply as indicated, doing all work orally:

1. 85.4×10.	**5.** $.0042\times 1000$	**9.** 2.364×10^2.
2. 4.003×100.	**6.** 24×100.	**10.** 64.30×10^3.
3. 42.75×10.	**7.** $.183\times 1000$.	**11.** 1.257×10^2.
4. $.00826\times 100$.	**8.** 250×10.	**12.** $.004286\times 10^3$.

153. Multiplying by a multiple of a power of 10. Since the numbers 20, 30, 40, etc., may be written 2×10, 3×10, 4×10, etc., we may multiply by these numbers by first multiplying by 10, as shown in §152, and then multiplying by 2, 3, 4, etc.

For example,
$$14.62\times 30 = 14.62\times 10\times 3 = 146.2\times 3 = 438.6$$

Similarly, we may multiply by 200, 400, 500, etc., by first changing these multipliers to 100×2, 100×4, 100×5, etc.

Exercises

Perform the following multiplications:

1. $.75\times 200$.	**4.** $.50\times 300$.	**7.** 2.03×400.
2. 1.44×30.	**5.** $.12\times 4000$.	**8.** 10.6×70.
3. $.24\times 2000$.	**6.** 293.5×60.	**9.** 9.12×200.

154. Arranging the factors in convenient order before multiplying. We have seen (§8) that the factors of a product, as $2\times 4\times 5$, may be arranged in any

order without changing the result, *i.e.*, the product $2\times4\times5=2\times5\times4=4\times5\times2$. Rearranging the order of the factors sometimes simplifies the work of multiplying. This is seen in the following illustrations:

1. $2\times685\times5=2\times5\times685=10\times685=6850$.
2. $4\times7\times25 \quad =4\times25\times7 \quad =100\times7 \quad =700$.
3. $250\times7\times2=250\times2\times7=500\times7 \quad =3500$.

Exercises

Arrange the factors in the products below in convenient order and state the result, doing as much as you can orally:

1. $50\times84\times2$.

Solution: $100\times84=8400$.

2. $25\times16\times4$.
3. $14\times10\times3$.
4. $5\times218\times3$.
5. $15\times8\times4$.
6. $5\times75\times6$.
7. $16\times5\times4$.
8. $50\times13\times2$.
9. $3\times16\times30$.
10. $8\times94\times5$.

155. How to multiply by a number differing by a small amount from a power of 10. The numbers 9, 99, 999, may be written $10-1$, $100-1$, $1000-1$. Similarly, 98, 998 may be written $100-2$, $1000-2$. This suggests a method of multiplying which is illustrated in the following examples:

1. 325×9. *Solution:*
$$\begin{array}{r} 10\times325=3250 \\ 1\times325=325 \\ \hline \text{Difference}=2925 \end{array}$$

2. 9.72×99. *Solution:*
$$\begin{array}{r} 100\times9.72=972 \\ 1\times9.72=9.72 \\ \hline \text{Difference}=962.28 \end{array}$$

3. 3.64×97. *Solution:*

$$\begin{array}{r} 100 \times 3.64 = 364 \\ 3 \times 3.64 = 10.92 \\ \hline \text{Difference} = 353.08 \end{array}$$

Exercises

Using the method above, find the following products:

1. 8.56×999.

Solution:

$$\begin{array}{r} 8560 \\ 8.56 \\ \hline 8551.44 \end{array}$$

2. 9×47.	**6.** 19×14.	**10.** 29×25.4	**14.** 996×235.
3. 9×32.	**7.** 99×25.1.	**11.** 39×1.38.	**15.** $997 \times .324$.
4. 9×81.	**8.** 99×16.5.	**12.** $59 \times .75$.	**16.** 998×15.7.
5. 19×25.	**9.** 99×14.2.	**13.** 98×1.23.	**17.** 999×25.

156. Multiplying by 5, 25, 50, 75, 125. Simple methods of multiplying by 5, 25, 50, 75, 125 are explained below.

1. Since $5 = \frac{10}{2}$, the product of a number by 5 may be found by first multiplying the number by 10 and then dividing the result by 2.

Thus, $3.42 \times 5 = \dfrac{3.42 \times 10}{2} = \dfrac{34.2}{2} = 17.1$.

Briefly, we proceed as follows:

$$3.42 \times 5 = \frac{34.2}{2} = 17.1$$

2. Since $25 = \frac{100}{4}$, the product of a number by 25

may be found by first multiplying by 100 and then dividing by 4.

Thus, $3.84 \times 25 = \frac{384}{4} = 96.$

3. Since $50 = \frac{100}{2}$, we may multiply by 50 by first multiplying by 100 and then dividing by 2.

Thus, $7.2 \times 50 = \frac{720}{2} = 360.$

4. Since $75 = \frac{3}{4} \times 100$, we may multiply by 75 by first multiplying by 100 and then by $\frac{3}{4}$.

Thus, $16 \times 75 = 1600 \times \frac{3}{4} = 400 \times 3 = 1200.$

5. Since $125 = \frac{1000}{8}$, we may multiply by 1000 and divide by 8.

Thus, $56 \times 125 = \frac{56000}{8} = 7000.$

Exercises

Perform the following multiplications, using the methods explained above:

1. 96×25.	6. 72×25.	11. 36×25.	16. 48×25.
2. 72×125.	7. 64×125.	12. 28×75.	17. 608×50.
3. 98×50.	8. 36×50.	13. 506×50.	18. 48×125.
4. 32×75.	9. 72×5.	14. 576×125.	19. 284×5.
5. 64×75.	10. 24×5.	15. 642×5.	20. 36×75.

157. How to multiply quickly by 11. The two examples below illustrate short processes of multiply-

ing by 11. The multiplier 11 is changed to $10+1$.

1. 236×11.

Solution: Multiplying by 10, we have 2360
Multiplying by 1, 236
Adding, 2596

From the results we derive the following:

To multiply 236 by 11, write first the right-hand digit, 6. Then, passing from right-hand digit to the left, write the sums of the adjacent digits as $6+3$ and $3+2$. Finally write the left-hand digit.

2. 2569×11.

Solution: Multiplying by 10, we have 25690
Multiplying by 1, 2569
Adding, 28259

As in example 1, the result may be obtained directly as follows:

Write the right-hand digit, 9.

Passing from the right-hand digit to the left, write the sums of the adjacent digits, carrying if necessary. Thus $9+6=15$. Write 5 and carry 1.

$6+5$ are 11 and 1 are 12. Write 2 and carry 1.
$5+2$ are 7 and 1 are 8. Write 8.
Finally, write the left-hand digit 2.

Exercises

Find the following products by the short method explained above:

1. 523×11.	5. 857×11.	9. 3285×11.
2. 708×11.	6. 423×11.	10. 4172×11.
3. 324×11.	7. 906×11.	11. 6498×11.
4. 134×11.	8. 637×11.	12. 28357×11.

158. Multiplying by parts of 100. We have previously learned how to change multiplication by a whole number, as 5, 50, 125, to multiplication by parts of 10, 100, or 1000. Goods to be sold are frequently marked at fractional prices which are even parts of 100. Articles are sold 2 pounds for 25c, or 3 for $1.00. Everybody should know the short methods for multiplying by such prices. They are explained in the following examples:

1. To multiply by $12\frac{1}{2}$, change $12\frac{1}{2}$ to $\frac{1}{8}\times 100$, *i.e.*, multiply first by 100 and then divide by 8.
2. To multiply by $33\frac{1}{3}$, change $33\frac{1}{3}$ to $\frac{1}{3}\times 100$, *i.e.*, multiply first by 100 and then divide by 3.
3. To multiply by $16\frac{2}{3}$, change $16\frac{2}{3}$ to $\frac{1}{6}\times 100$, *i.e.*, multiply first by 100 and then divide by 6.

The table below gives a list of multipliers and their equivalents which are commonly used. They will be found very helpful in problems which involve buying and selling articles. For example: John's father bought 5 collars that were sold 3 for $1. How much did he pay for them? This problem may be solved as follows:

$$5\times 33\tfrac{1}{3}=\frac{500}{3}=166\tfrac{2}{3}$$

He paid $1.67.

$12\frac{1}{2}=\frac{1}{8}\times 100$	$.12\frac{1}{2}=\frac{1}{8}$	$10\%=\frac{1}{10}$
$33\frac{1}{3}=\frac{1}{3}\times 100$	$.33\frac{1}{3}=\frac{1}{3}$	$20\%=\frac{1}{5}$
$16\frac{2}{3}=\frac{1}{6}\times 100$	$.16\frac{2}{3}=\frac{1}{6}$	$25\%=\frac{1}{4}$
$66\frac{2}{3}=\frac{2}{3}\times 100$	$.66\frac{2}{3}=\frac{2}{3}$	$50\%=\frac{1}{2}$
$62\frac{1}{2}=\frac{5}{8}\times 100$	$.62\frac{1}{2}=\frac{5}{8}$	$60\%=\frac{3}{5}$
$37\frac{1}{2}=\frac{3}{8}\times 100$	$.37\frac{1}{2}=\frac{3}{8}$	$75\%=\frac{3}{4}$
$87\frac{1}{2}=\frac{7}{8}\times 100$	$.87\frac{1}{2}=\frac{7}{8}$	$80\%=\frac{4}{5}$

Exercises

Using the table above, find the following products by the short method:

1. $64\times12\frac{1}{2}$.	8. $56\times.62\frac{1}{2}$.	15. 20% of 75.
2. $18\times66\frac{2}{3}$.	9. $72\times.87\frac{1}{2}$.	16. 75% of 72.
3. $96\times37\frac{1}{2}$.	10. $128\times.12\frac{1}{2}$.	17. 25% of 68.
4. $32\times62\frac{1}{2}$.	11. $252\times.66\frac{2}{3}$.	18. 10% of 85.
5. $45\times33\frac{1}{3}$.	12. $14\times.37\frac{1}{2}$.	19. 80% of 85.
6. $88\times87\frac{1}{2}$.	13. $216\times.33\frac{1}{3}$.	20. 50% of 98.
7. $36\times16\frac{2}{3}$.	14. $246\times.16\frac{2}{3}$.	21. 30% of 57.

159. Finding the square of a number which ends in $\frac{1}{2}$, .5, or 5. Show that $6\frac{1}{2}\times6\frac{1}{2}=(6+\frac{1}{2})(6+\frac{1}{2})$

$$=6^2+2\times6\times\frac{1}{2}+(\frac{1}{2})^2=6^2+6\times1+(\frac{1}{2})^2$$

$=6(6+1)+(\frac{1}{2})^2=42+\frac{1}{4}=42\frac{1}{4}$. To obtain the result, briefly multiply the 6 of $6\frac{1}{2}$ by the consecutive integer 7 which gives 42. Then add the square of $\frac{1}{2}$.

Similarly, $6.5\times6.5=6\times7+(.5)^2$

$$=42.25$$

To multiply 65×65 proceed as with 6.5, leaving out the decimal point.

Exercises

1. $3\frac{1}{2}\times3\frac{1}{2}$.	4. $6\frac{1}{2}\times6\frac{1}{2}$.	7. 9.5×9.5.	10. 45×45.
2. $4\frac{1}{2}\times4\frac{1}{2}$.	5. $7\frac{1}{2}\times7\frac{1}{2}$.	8. 12.5×12.5.	11. 75×75.
3. $5\frac{1}{2}\times5\frac{1}{2}$.	6. $11\frac{1}{2}\times11\frac{1}{2}$.	9. 8.5×8.5.	12. 85×85.

160. A quick way of finding the product of two mixed numbers. The product of $6\frac{3}{7}$ by $4\frac{2}{5}$ may be found in two ways:

1. Write $6\frac{3}{7}\times 4\frac{2}{5}=(6+\frac{3}{7})(4+\frac{2}{5})$ and multiply the two binomials, arranging the work as follows:

$$\begin{array}{rl} & 6\frac{3}{7} \\ & \underline{4\frac{2}{5}} \\ 6\times 4 = & 24 \\ 6\times\frac{2}{5} = & 2\frac{2}{5} \\ 4\times\frac{3}{7} = & 1\frac{5}{7} \\ \frac{3}{7}\times\frac{2}{5} = & \frac{6}{35} \\ \hline & 27\frac{45}{35}=28\frac{2}{7} \end{array}$$

The method should be used when the numbers are large.

2. Change the mixed numbers to fractions:

$6\frac{3}{7}\times 4\frac{2}{5}=\dfrac{45\times 22}{7\times 5}=\dfrac{9\times 22}{7}=\dfrac{198}{7}=28\frac{2}{7}$. This method is to be used when the numbers are small.

Exercises

Find the following products:

1. $4\frac{2}{3}\times 6\frac{1}{2}$.	4. $15\frac{4}{5}\times 2\frac{1}{2}$.	7. $24\frac{1}{2}\times 9\frac{1}{4}$.	10. $15\frac{1}{3}\times 5\frac{1}{2}$.
2. $8\frac{4}{5}\times 5\frac{3}{4}$.	5. $8\frac{1}{3}\times 9\frac{3}{8}$.	8. $18\frac{3}{4}\times 7\frac{1}{3}$.	11. $22\frac{1}{4}\times 6\frac{2}{3}$.
3. $16\frac{1}{4}\times 6\frac{1}{2}$.	6. $7\frac{1}{2}\times 8\frac{1}{3}$.	9. $25\frac{1}{8}\times 8\frac{2}{3}$.	12. $24\frac{1}{2}\times 7\frac{1}{4}$.

161. How to multiply two two-figure numbers. Since $38\times 26=(30+8)(20+6)$, we may multiply the two numbers like two binomials. Thus,

$$\begin{array}{r} 6\times\ 8=48 \\ 6\times 30=180 \\ 20\times\ 8=160 \\ 20\times 30=600 \\ \hline \therefore 38\times 26=988 \end{array}$$

Briefly, we may write the product 988 as follows:

$6 \times 8 = 48$. Write 8 and carry 4.

$4 + (6 \times 3) + (2 \times 8) = 38$. Write 8 and carry 3.

$3 + (2 \times 3) = 9$. Write 9.

Exercises

Write the following products, doing all the work orally.

1. 29×76.	**5.** 18×35.	**9.** 61×64.	**13.** 69×95.
2. 84×43.	**6.** 75×82.	**10.** 45×38.	**14.** 26×28.
3. 36×92.	**7.** 53×59.	**11.** 87×12.	**15.** 34×79.
4. 72×25.	**8.** 97×94.	**12.** 52×46.	**16.** 13×36.

162. A quick way of finding successive discounts. To take two successive discounts, as 15% and 10%, we may proceed as follows: Suppose we wish to take two successive discounts of 900.

1. The given sum $= 900$

$$\begin{array}{ll} 15\% \text{ of } 900 & = 135 \\ \hline \text{Difference} & = 765 \\ 10\% \text{ of } 765 & = 76.5 \\ \hline \text{Difference} & = 688.5 \end{array}$$

2. The same result may be obtained by first adding the two discounts and then subtracting the product. Thus, $15\% + 10\% = 25\%$

$$\begin{array}{l} 15\% \times 10\% = 1.5\% \\ \hline \text{Difference} = \ 23.5\% \end{array}$$

$\therefore 23.5\%$ of $900 = 211.5$

$\$900 - \$211.5 = \$688.5$.

Exercises

Find the following successive discounts:

Sums	$460	$850	$264	$740	$375
Discounts	20%, 10%	40%, 15%	15%, 15%	10%, 5%	60%, 20%

Division

163. Tests of divisibility. In changing a fraction, as $\frac{42}{72}$, to lowest terms we divide numerator and denominator by the same factor. It is therefore necessary to be able to find the factors of the numbers 42 and 72. The following rules help us to tell easily some of the factors most commonly found in numbers.

1. *A number ending in 5 or 0 is divisible by 5. A number ending in 0 is divisible by 5 and 10.*

2. *Even numbers are divisible by 2.*

3. *A number is divisible by 4 if the two-figure number formed by the ten and unit digits is divisible by 4.*

The reason for the third rule may be seen as follows: Write the number 4632 in the form 4000+600+32. Since the numbers 1000 and 100 are divisible by 4, the number 4600 is also divisible by 4. Therefore if the number formed by the last two digits is divisible by 4, the whole number must be divisible by 4.

Which of the numbers below are divisible by 4?

348, 562, 491, 872, 4180.

4. *A number is divisible by 8 if the number formed by the last three digits is divisible by 8.*

This may be shown as in the preceding case. Since 1000 or any higher power of 10 is divisible by 8, it follows that a number is divisible by 8 if the number formed by the last three digits is divisible by 8.

5. *A number is divisible by 3 or 9 if the sum of the digits is divisible by 3 or 9.*

To show this, write a three-figure number in the form $100a+10b+c$. Dividing by 3 we have

$$\frac{100a+10b+c}{3}=33\tfrac{1}{3}a+3\tfrac{1}{3}b+\tfrac{1}{3}c.$$
$$=33a+3b+\tfrac{1}{3}(a+b+c)$$

which is a whole number if the sum $a+b+c$ is divisible by 3.

Exercises

Test the following numbers for divisibility by 2, 3, 4, 5, 8, 9, and 10:

1.	64	642	523	465	428	641	460
2.	78	231	274	406	693	624	342
3.	92	325	724	827	852	412	784
4.	65	910	947	215	426	318	386
5.	72	736	211	712	232	945	364
6.	86	912	512	390	825	432	625
7.	70	603	275	326	924	306	280

8. Make tests of divisibility by 6; 12; 15; 18.

164. Simplifying division by dividing common factors into dividend and divisor. Long division may often be simplified or changed to short division by dividing

out common factors. Thus, in $738 \div 15$ we may divide the factor 3 into dividend and divisor, changing the problem to $246 \div 5$.

Exercises

Simplify the following division exercises:

1. $624 \div 16$.	**4.** $888 \div 24$.	**7.** $7520 \div 48$.	**10.** $1629 \div 36$.
2. $512 \div 32$.	**5.** $2556 \div 63$.	**8.** $3405 \div 25$.	**11.** $3048 \div 56$.
3. $837 \div 39$.	**6.** $4488 \div 27$.	**9.** $4086 \div 30$.	**12.** $5640 \div 54$.

165. Division by a power of 10. A number may be divided by 10, 100, or 1000 by moving the decimal point one, two or three places to the left.

Thus, $3614 \div 10 = 361.4$; $361.4 \div 100 = 3.614$; $3.614 \div 1000 = .3614$; $.3614 \div 10 = .03614$.

A number may be divided by .1, .01, .001, by multiplying the number by 10, 100, or 1000. Thus, $84.62 \div .1 = 84.62 \times 10 = 846.2$.

When the divisor ends in one or several zeros, the zeros may be left off and the decimal point in the dividend moved to the left one place for each zero left off. Thus, $5139 \div 80 = 513.9 \div 8$.

Exercises

Divide in the simplest way:

1. $54 \div 10$.	**4.** $76 \div .1$.	**7.** $524 \div 40$.
2. $8.6 \div 100$.	**5.** $94 \div .01$.	**8.** $123 \div 300$.
3. $22.3 \div 1000$.	**6.** $82 \div .001$.	**9.** $350 \div 500$.

166. How to divide by 5, 25, and 125. Since $5 = \frac{10}{2}$, any number divided by 5 may be divided by $\frac{10}{2}$, or

multiplied by $\frac{2}{10}$. Hence, to divide a number by 5 multiply it by 2 and divide the result by 10.

Similarly, since $25 = \frac{100}{4}$, to divide by 25 multiply by 4 and divide by 100. Since $125 = \frac{1000}{8}$, we may divide by 125 by multiplying by 8 and dividing by 1000.

Exercises

Divide as indicated, doing all you can orally:

1. $715 \div 5$.	4. $2525 \div 25$.	7. $1670 \div 25$.	10. $7255 \div 125$.
2. $1580 \div 125$.	5. $225 \div 5$.	8. $8575 \div 125$.	11. $2565 \div 25$.
3. $2135 \div 25$.	6. $1325 \div 125$.	9. $175 \div 5$.	12. $3135 \div 5$.

167. How to divide by $12\frac{1}{2}$, $16\frac{2}{3}$, $33\frac{1}{3}$. These three numbers divide into 100 without remainder, and are equal to $\frac{100}{8}$, $\frac{100}{6}$, and $\frac{100}{3}$. Hence to divide by them, we may multiply by 8, 6, and 3, respectively, and then divide by 100.

Exercises

Divide as indicated, doing all you can orally:

1. $96 \div 12\frac{1}{2}$.	4. $364 \div 16\frac{2}{3}$.	7. $3.68 \div 16\frac{2}{3}$.
2. $98 \div 33\frac{1}{3}$.	5. $250 \div 12\frac{1}{2}$.	8. $184 \div 33\frac{1}{3}$.
3. $84 \div 16\frac{2}{3}$.	6. $356 \div 12\frac{1}{2}$.	9. $84.2 \div 12\frac{1}{2}$.

168. A short way of dividing by $3\frac{1}{8}$, $6\frac{1}{4}$, $62\frac{1}{2}$, $87\frac{1}{2}$. From the facts that $3\frac{1}{8}=\frac{25}{8}=\frac{100}{32}$, $6\frac{1}{4}=\frac{25}{4}=\frac{100}{16}$, $62\frac{1}{2}=\frac{125}{2}=\frac{1000}{16}$, $87\frac{1}{2}=\frac{175}{2}=\frac{700}{8}$ it is easy to derive rules for dividing by these numbers. Formulate the rules and use them in the following exercises:

Exercises

1. $46 \div 3\frac{1}{8}$.	4. $14 \div 6\frac{1}{4}$.	7. $12 \div 6\frac{1}{4}$.	10. $61 \div 3\frac{1}{8}$.
2. $22 \div 6\frac{1}{4}$.	5. $21 \div 3\frac{1}{8}$.	8. $14 \div 87\frac{1}{2}$.	11. $28 \div 87\frac{1}{2}$.
3. $21 \div 87\frac{1}{2}$.	6. $18 \div 62\frac{1}{2}$.	9. $16 \div 3\frac{1}{8}$.	12. $36 \div 6\frac{1}{4}$.

Square Root

169. A short method of approximating the square root of a number. For practical purposes it is often sufficient to find the approximate value of the square root of a number to one or two decimal places only. The following examples explain the method:

1. Find the square root of 87.

a. First approximation: Since $9^2=81$, and $10^2=100$, the square root of 87 lies a little less than midway between 9 and 10. Let us suppose it to be 9.4.

b. Divide 87 by the first approximation, 9.4. This gives the first quotient 9.25.

c. Second approximation: Find the average of the first approximation 9.4 and the quotient 9.25. This is 9.31.

d. Divide 87 by the second approximation, 9.31. This gives the second quotient, 9.34.

e. Third approximation: Find the average of the second approximation and the second quotient. This is 9.315.

f. Continue the process as far as it seems desirable.

2. Find the square root of 1′49′06.

a. The square root lies between 120 and 130. Let us suppose it to be 123.

b. Divide 14906 by 123. The quotient is 121 approximately.

c. Find the average of 123 and 121. This is 122.

d. Divide 14906 by 122. This gives 122.1, the approximate square root of 14906.

Exercises

Find the approximate square root of each of the following numbers:

1. 34.	**5.** 9215.	**9.** 16312.
2. 23.	**6.** 5202.	**10.** 25484.
3. 56.	**7.** 1525.	**11.** 91324.
4. 98.	**8.** 9147.	**12.** 76518.

Interest

170. How to compute interest with interest tables. Bankers, loan and insurance agents, and other business men who compute interest frequently and wish to do it quickly and accurately, use *interest tables*. The tables are usually bound into book form and so arranged that interest can be computed for various amounts, days, and rates. The method of finding interest by means of a table can be understood by using the short table following.

6% Simple Interest Table

Days	$100	$200	$300	$400	$500
1	0.0167	0.0333	0.0500	0.0667	0.0833
2	0.0333	0.0667	0.1000	0.1333	0.1667
3	0.0500	0.1000	0.1500	0.2000	0.2500
4	0.0667	0.1333	0.2000	0.2667	0.3333
5	0.0833	0.1667	0.2500	0.3333	0.4167
6	0.1000	0.2000	0.3000	0.4000	0.5000
7	0.1167	0.2333	0.3500	0.4667	0.5833
8	0.1333	0.2667	0.4000	0.5333	0.6667
9	0.1500	0.3000	0.4500	0.6000	0.7500
10	0.1667	0.3333	0.5000	0.6667	0.8333
11	0.1833	0.3667	0.5500	0.7333	0.9167
12	0.2000	0.4000	0.6000	0.8000	1.0000
13	0.2167	0.4333	0.6500	0.8667	1.0833
14	0.2333	0.4667	0.7000	0.9333	1.1667
15	0.2500	0.5000	0.7500	1.0000	1.2500
16	0.2667	0.5333	0.8000	1.0667	1.3333
17	0.2833	0.5667	0.8500	1.1333	1.4167
18	0.3000	0.6000	0.9000	1.2000	1.5000
19	0.3167	0.6333	0.9500	1.2667	1.5833
20	0.3333	0.6667	1.0000	1.3333	1.6667
21	0.3500	0.7000	1.0500	1.4000	1.7500
22	0.3667	0.7333	1.1000	1.4667	1.8333
23	0.3833	0.7667	1.1500	1.5333	1.9167
24	0.4000	0.8000	1.2000	1.6000	2.0000
25	0.4167	0.8333	1.2500	1.6667	2.0833

Days	$100	$200	$300	$400	$500
26	0.4333	0.8667	1.3000	1.7333	2.1667
27	0.4500	0.9000	1.3500	1.8000	2.2500
28	0.4667	0.9333	1.4000	1.8667	2.3333
29	0.4833	0.9667	1.4500	1.9333	2.4167
30	0.5000	1.0000	1.5000	2.0000	2.5000

Days	$600	$700	$800	$900	$1000
1	0.1000	0.1167	0.1333	0.1500	0.167
2	0.2000	0.2333	0.2667	0.3000	0.333
3	0.3000	0.3500	0.4000	0.4500	0.500
4	0.4000	0 4667	0.5333	0.6000	0.667
5	0.5000	0.5833	0.6667	0.7500	0.833
6	0.6000	0.7000	0.8000	0.9000	1.000
7	0.7000	0.8167	0.9333	1.0500	1.167
8	0.8000	0.9333	1.0667	1.2000	1.333
9	0.9000	1.0500	1.2000	1.3500	1.500
10	1.0000	1.1667	1.3333	1.5000	1.667
11	1.1000	1.2833	1.4667	1.6500	1.833
12	1.2000	1.4000	1.6000	1.8000	2.000
13	1.3000	1.5167	1.7333	1.9500	2.167
14	1.4000	1.6333	1.8667	2.1000	2.333
15	1.5000	1.7500	2.0000	2.2500	2.500
16	1.6000	1.8667	2.1333	2.4000	2.667
17	1.7000	1.9833	2.2667	2.5500	2.833
18	1.8000	2.1000	2.4000	2.7000	3.000
19	1.9000	2.2167	2.5333	2.8500	3.167
20	2.0000	2.3333	2.6667	3.0000	3.333

Days	$600	$700	$800	$900	$1000
21	2.1000	2.4500	2.8000	3.1500	3.500
22	2.2000	2.5667	2.9333	3.3000	3.662
23	2,3000	2.6833	3.0667	3.4500	3.833
24	2.4000	2.8000	3.2000	3.6000	4.000
25	2.5000	2.9167	3.3333	3.7500	4.167
26	2.6000	3.0333	3.4667	3.9000	4.333
27	2.7000	3.1500	3.6000	4.0500	4 500
28	2.8000	3.2667	3.7333	4.2000	4.667
29	2.9000	3.3833	3.8667	4.3500	4.833
30	3.0000	3.5000	4.0000	4.5000	5.000

Exercises

1. From the interest table find the interest on $400 at 6% for 15 days; on $700 for 28 days; on $500 for 21 days.

2. Find the interest on $350 for 15 days.

Solution: Interest on $300 for 15 days = $0.7500
Interest on $½(100) for 15 days = 0.1250

Total = $0.8750 or $0.88

3. Find the interest at 6% on the following:

$450 for 20 days.	$1000 for 18 days.
$350 for 22 days.	$3000 for 16 days.
$850 for 12 days.	$3550 for 27 days.

4. Find the interest at 6% on the following:

$80 for 10 days.	$75 for 20 days.
$60 for 24 days.	$83 for 18 days.
$68 for 13 days.	$64 for 27 days.

5. Find the interest at 6% on the following:

\$2360 for 30 days.	\$2530 for 1 year 360 days.
\$1640 for 20 days.	\$1350 for 1 year 5 months.
\$350 for 90 days.	\$1275 for 2 years 8 months.

6. Using the 6% table find the interest at 2% and at 3% on each of the following:

\$2500 for 17 days.	\$3680 for 26 days.
\$8465 for 21 days.	\$9360 for 27 days.

7. Explain how the 6% table may be used to compute interest at 1%, 4%, 5%, 8%.

CHAPTER XI

SUPPLEMENTARY EXERCISES

The Fundamental Operations

171. Practice and review. In Book Two are taught many important algebraic processes. To help you gain the skill necessary to perform them with ease, this chapter provides numerous exercises. They may be worked in addition to the problems given in the text whenever there seems need for practice, or they may be used to help you review your work from time to time.

172. Combining similar terms. In the following exercises perform the indicated operations and change each result to the simplest form:

Exercises

1. $4x+3x+2x$.
2. $2m+5m+m$.
3. $16a+3ab-5a$.
4. $10b-6b+3b$.
5. $2t+\frac{1}{2}t+\frac{3}{4}t$.
6. $\frac{1}{2}x-\frac{3}{4}x+\frac{5}{8}x$.
7. $\frac{3x}{5}+\frac{x}{6}-\frac{2x}{3}$.
8. $\frac{x}{3}-\frac{2x}{7}-\frac{6x}{5}$.
9. $5w+7-6w+4+12w$.
10. $7x-3-12x+7-12$.
11. $3p-8-5p+4-4p$.
12. $6p-3p+5+2-8p-10$.
13. $3x^2+5+7x^2-8-4x^2$.
14. $6a^3-3b+10a^3+5b-16a^3$.
15. $3m^4-8t-7m^4-3t+4m^4$.
16. $5a-3t^5+4a-7a+4t^5-7t^5$.
17. $-4xy-2z+6xy+8z+3xy-z$.
18. $8x+(+4x)+(-3x)-(10x)$.
19. $14a-(2b)-(-3b)+(-7a)-8$.
20. $-3m-(4a)+(5m)+(-2m)+6a$.

173. Finding the values of algebraic expressions. Letting $a=3$, $b=2$, $e=1.5$, $d=1$, and $x=-2$, find the value of each of the following:

EXERCISES

1. $2a-b$.
2. $5c-8d$.
3. a^2+ax.
4. $2ad-3bx$.
5. $3ab+ab^2$.
6. b^2-4ac.
7. $2ac+ac^2$.
8. $6d^2+3a^2b$.
9. πd.
10. $\frac{1}{2}ab$.
11. $\frac{3}{4}\pi a^3$.
12. $\pi a^2 b$.
13. $\frac{1}{2}a(b+c)$.
14. $\pi a^2 x$.
15. $\dfrac{abc}{100}$.
16. $\frac{1}{2}ax^2$.
17. $2ab+2ac+2bc$.
18. $\frac{1}{2}a+\frac{1}{3}b-\frac{1}{4}c$.
19. $3.4b-1.4a+6.8d$.
20. $2\frac{1}{2}x-4\frac{1}{3}a-5\frac{1}{4}d$.
21. $3.1c+6.5a-10.1b$.
22. $4\frac{1}{2}x-3\frac{1}{3}b+5\frac{1}{3}d$.
23. $3.4a-3.1b+4.2c$.
24. $5.3a+5.6b-3.2x$.
25. a^3+3a^2-6a-4.
26. $3x^3-5x^2+x-6$.
27. $2a^2-3b^2+8c-d$.

174. Addition and subtraction of polynomials. Add or subtract as indicated and combine similar terms:

EXERCISES

1. $9a-(8+2a)$.
2. $-8n+(5-4n)$.
3. $3a-(2a+6)$.
4. $(5-4x)-8x$.
5. $2m-(3n-6m)$.
6. $3r+(5r+4t)$.
7. $(6g-(3g+4h)$.
8. $2a+(-4a+6b)$.
9. $2b-(b-2a)$.
10. $(5x-y)+(7x-2y)$.
11. $(8a-9b)-(4a+5b)$.
12. $(3\frac{1}{2}a+4\frac{1}{3}b)+(4\frac{1}{2}a-3\frac{2}{3}b)$.

13. $(5m-10n)-(m+2n)$.
14. $(a+b+c)-(a-b-c)$.
15. $(2x^2+x-4)+(3x^2+4x)$.
16. $(5c-d)-[8+(12c-6d)]$.
17. $(a-2b)-[(1+3a)-8b]$.
18. $(8b-6f)+[7b-(3c-f)]$.
19. $4-\frac{2-3a}{2}+3a$.
20. $3x-\frac{4-2x}{5}-\frac{x+3}{2}$.

175. Multiplication of monomials. Multiply as indicated and change all results to the simplest form:

Exercises

1. $3\cdot 5x$.
2. $6a^2\cdot 3b$.
3. $2a\cdot 7ab^2$.
4. $3c(-4ac^2)$.
5. $5a\cdot 2a^2bc$.
6. $(6m^2)(-3mn)$.
7. $(m^2)^3$.
8. $3(-p^2)^2$.
9. $(ab^2)^3$.
10. $(-2mn^2)^3$.
11. $2x\cdot 16xy^2$.
12. $(-3a)\left(\frac{2b}{3a}\right)$.
13. $(-\frac{1}{2}xy)(6x^2y^2)$.
14. $4a(-5ab^2)$.
15. $10a(3x)(-2ax)$.
16. $ab(-3b^2)(2ab)$.
17. $ab(-3b^2)(2ab)$.
18. $a(b^2)(-2a^2b)$.
19. $\left(\frac{7x}{12}\right)\left(-\frac{20}{21x^2}\right)$.
20. $3x\left(-\frac{2}{5x}\right)\left(\frac{10}{3x^2}\right)$.

176. Multiplication of a polynomial by a monomial. In the following exercises carry out the multiplications and combine similar terms:

EXERCISES

1. $a(3x+2)$.
2. $-b(b+4c)$.
3. $\frac{1}{2}x(x+4)$.
4. $3p(4a-b)$.
5. $-ab(3a^2-ab)$.
6. $\frac{c}{6}(12a+6b)$.
7. $3m\left(\frac{m}{6}+\frac{n}{9}\right)$.
8. $2x(3y-4)+5(x+3)$.
9. $-y(3y^2-4y+6)$.
10. $(a+b)m+(x+y)n$.
11. $x(x^2+bx)-a(a^2-x)$.
12. $a(a^2-4a+3)-5a^2$.
13. $9x(2x^2-x-1)+16x$.
14. $\frac{m}{3}(6x-2)-\frac{m}{2}(4x-1)$.

177. Multiplication of a polynomial by a polynomial. Multiply as indicated and change each product to the simplest form:

EXERCISES

1. $(x-3y)^2$.
2. $(x+7)(x-3)$.
3. $(2a+6)(a-4)$.
4. $(6x-1)(3x+2)$.
5. $(2a-b+3c)^2$.
6. $(\frac{3m}{2}-4)(6m+2)$.
7. $(a+2x)(x-a)$.
8. $(2r+1)(r^3-7r)$.
9. $(2a-5)(a^2+3a-2)$.
10. $(\frac{1}{2}a+\frac{1}{3}b)(12a-6b+18c)$.
11. $(3x^2-2x+4)(5x-3)$.
12. $(m^2-mn+n^2)(m-n)$.
13. $(x^3+x-4-3x^2)(x^2+x)$.
14. $(a+b)(a^2-ab+b^2)$.
15. $(x-y)(x^2-xy+y^2)$.
16. $(a^3-3)(a^3+a^2-a+1)$.
17. $(m^2-2)(m^3+2m^2-6m+4)$.
18. $(2a^2-b+c)(a-2b-c)$.
19. $(x^2+xy+y^2)(x^2+xy-y^2)$.
20. $(\frac{2}{3}a^2+\frac{1}{2}a+\frac{1}{4})(12a^2+6a+24)$.

The Formula

178. Solving formulas. Solve each of the following formulas for the required literal number:

Exercises

1. $i=\frac{prt}{100}$. Find t.
2. $A=\pi r^2$. Find r.
3. $V=abc$. Find c.
4. $V=\frac{4}{3}\pi r^3$. Find r.
5. $L=\pi rs$. Find s.
6. $V=\frac{1}{3}\pi r^2h$. Find r.
7. $C=\frac{E}{R}$. Find R.
8. $s=\frac{w}{L}$. Find w.
9. $M=\frac{SI}{T}$. Find I.
10. $K=\frac{Mv^2}{2}$. Find v.
11. $C=\frac{E}{R+r}$. Find R.
12. $A=\frac{1}{2}h(a+b)$. Find a.

179. Finding the values of formulas. In the following exercises find the values of the unknown numbers:

Exercises

1. $p=20a$; $a=4.23$.
2. $l=2ab+2ac$; $a=5$, $b=\frac{3}{2}$, $c=4$.
3. $c=2\pi r$; $c=98$.
4. $A=\pi r^2$; $A=50$.
5. $A=\frac{1}{2}bh$; $A=20$, $b=3\frac{1}{2}$.
6. $A=\frac{1}{2}h(a+b)$; $A=120$, $h=6$, $a=2$.
7. $V=\pi r^2h$; $V=140$, $h=10$.
8. $L=2\pi rh$; $L=60$, $h=8$.
9. $V=\frac{1}{3}bh$; $V=27$, $h=6$.
10. $F=32+\frac{9}{5}C$; $F=34$.
11. $s=\frac{1}{2}gt^2$; $s=110$, $g=32$.
12. $A=P+PRT$; $A=200$, $P=180$, $T=2$.

THE EQUATION

180. Linear equations with one unknown. Solve the following equations:

EXERCISES

1. $2x=8.$
2. $12x+4x=-15+35.$
3. $.5m=.2m+1.2.$
4. $4x=6x+10.$
5. $.4x+.3=.5-.7x.$
6. $7a-13a=18-42.$
7. $12a-10-15a=7.$
8. $13m+18=-9m-17.$
9. $9x-7=-13x+6.$
10. $2(x+3)=21-x.$
11. $-3(9a+17)=-14a-7.$
12. $a-2(4-5a)=14.$
13. $11(13m-4)=-9m+35.$
14. $2a-20=3(2a-5).$
15. $6(a-3)-4(a+2)=-a+4.$
16. $4(2x-5)+15=3(x+10).$
17. $2(m-3)+3(m-2)=8.$
18. $8(11a-4)=7(13a-3).$

181. Linear equations with two unknowns. Solve the following equations by algebraic methods:

EXERCISES

1. $3m+n=5$
   $2m+n=2.$
2. $11x-7y=15$
   $2x+7y=11.$
3. $x+6y=-3$
   $2x-3y=9.$
4. $2x+y=10$
   $3x+2y=1.$
5. $5x-2y=0$
   $3x+5y=13.$
6. $4x+5y=76$
   $5x+4y=13.$
7. $3x=2y+10$
   $7x+3y=31.$
8. $7a+2b=8$
   $8a+3b=9.$

182. Quadratic equations in one unknown. Solve the following quadratic equations:

Exercises

1. $a^2=144.$
2. $2b^2=64.$
3. $x^2-36=0.$
4. $11x^2-99=0.$
5. $\frac{1}{2}gt^2=100.$
6. $x^2+64=100.$
7. $20a^2+8=28.$
8. $2x^2-125=-27.$
9. $a^2-4a-45=0.$
10. $p^2-5p-40=0.$
11. $x^2+9x=10.$
12. $a^2-5a=50.$
13. $a^2+5=6a.$
14. $2x^2+5x+2=0.$
15. $5a^2=2a+7.$
16. $9x^2-6x-4=0.$

183. Fractional equations. Change each of the equations below to the simplest form and solve:

Exercises

1. $\dfrac{x+8}{5}=12.$
2. $\dfrac{2+x}{5}=\dfrac{2-x}{2}.$
3. $\frac{5}{3}a=\frac{2}{3}a+2.$
4. $\dfrac{11a-4}{7}=\dfrac{13a-3}{8}.$
5. $\dfrac{4a+2}{3}-\dfrac{a-3}{4}=0.$
6. $\dfrac{7a-8}{7}-\dfrac{a+6}{2}=0.$
7. $\dfrac{4x-5}{2}-\dfrac{3x-3}{4}-3=0.$
8. $\dfrac{m-6}{m-2}=\dfrac{m+4}{m+5}.$
9. $\dfrac{a-6}{a-2}=\dfrac{a+3}{a+4}$
10. $\dfrac{x+3}{x-4}=\dfrac{x+2}{x-7}.$
11. $\dfrac{y+5}{y-4}=\dfrac{y+3}{y-2}.$
12. $\dfrac{a}{2}+\dfrac{3b}{5}=-8.$
    $6a-5b=26.$
13. $\dfrac{5y}{6}+\dfrac{z}{4}=7.$
14. $\dfrac{2y}{3}-\dfrac{z}{8}=3.$

184. Adding, subtracting, multiplying, dividing, and reducing fractions. Change each of the following expressions to the simplest form:

Exercises

1. $\frac{2}{7}+\frac{4}{7}$.

2. $\frac{a}{c}+\frac{b}{c}$.

3. $\frac{6a}{5b}-\frac{5a+7b}{5b}$.

4. $\frac{x}{4}-\frac{2x}{3}+\frac{x}{2}$.

5. $\frac{a}{3x}-\frac{5}{2x}$.

6. $\frac{a+b}{c}+\frac{a-b}{2c}$.

7. $\frac{a}{x}+b$.

8. $\frac{c}{y}+a$.

9. $\frac{7x^2}{3y^2}\cdot\frac{15y}{-x}$.

10. $\frac{2a^2x}{3b^2y}\cdot-\frac{5b}{4ax^2}$.

11. $\frac{16a^2b^2}{-27x^3y^2}\div\frac{8ab^2}{9x^2y}$.

12. $\frac{20xy^3}{-21a^4c}\div\frac{4x^2y}{3ac^2}$.

13. $\frac{9abc-18a^2b}{-9ab}$

14. $\frac{8a^2-20ab}{4a}$.

15. $\frac{5x^2y-15xy^2}{-5xy}$.

16. $\frac{9xy^2+6xy^2+3xy^2}{3xy}$.

17. $\frac{6ma+6mb}{3mc}$.

18. $\frac{35a^2b^2-49ab^3}{7ab^2}$.

19. $\frac{-a^4b^2-a^2b^2+5a^2b^3}{-a^2}$.

20. $\frac{-12m^3n+4m^2n^2}{-4m}$.

TABLES AND FORMULAS

Measure of Length

12 inches (in.)	= 1 foot (ft.)
3 feet	= 1 yard (yd.)
$5\frac{1}{2}$ yards or $16\frac{1}{2}$ feet	= 1 rod (rd.)
320 rods	= 1 mile (mi.)
1 mi.	= 320 rods = 1,760 yards = 5,280 feet = 63,360 inches

Measures of Surface

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.)
160 square rods	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)
36 square miles	= a township (tp.)
An acre	= a square approximately 209 feet on one side, or 4840 square yards, or 43,560 square feet

Measures of Volume

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.)
27 cubic feet	= 1 cubic yard (cu. yd.)
16 cubic feet	= 1 cord foot (cd. ft.)
128 cubic feet	= 1 cord (cd.)
1 board foot	= 1 foot long, 1 foot wide, and 1 inch thick

Equivalents

1 bushel	= $\frac{5}{4}$ cubic feet, or 2150.42 cubic inches
1 gallon	= 231 cubic inches
1 cubic foot of water	= 62.5 pounds (approx.)
1 ton of hay	= 500 cubic feet (approx.)
1 ton of hard coal	= 35 cubic feet (approx.)
1 ton of soft coal	= 38 cubic feet (approx.)

Liquid Measure

4 gills (gi.)	= 1 pint (pt.)
2 pints	= 1 quart (qt.)
4 quarts	= 1 gallon (gal.)
231 cubic inches	= 1 gallon
1 cubic foot	= 7.48 gallons
$31\frac{1}{2}$ gallons	= 1 barrel (bbl.)
4.27 cubic feet	= 1 barrel

Dry Measure

2 pints (pt.)	= 1 quart (qt.)
8 quarts	= 1 peck (pk.)
4 pecks	= 1 bushel (bu.)
$1\frac{1}{4}$ cubic feet	= 1 bushel (approx.)
1 bushel wheat or potatoes	= 60 pounds
1 bushel corn or rye	= 56 pounds
1 bushel oats	= 32 pounds
1 bushel barley	= 48 pounds
1 barrel of flour	= 196 pounds

Standard Weights

16 ounces (oz.)	= 1 pound (lb.)
100 pounds (lb.)	= 1 hundredweight (cwt.)
2000 pounds	= 1 ton
2240 pounds	= 1 long ton

Measures of Angles

60 seconds ($''$)	= 1 minute ($'$)
60 minutes	= 1 degree ($^\circ$)
360 degrees	= 4 right angles
90 degrees of angle	= 1 right angle

Measures of Arcs

90 degrees of arc.	= 1 quadrant
360 degrees of arc.	= 1 circumference

Paper Measure

24 sheets = 1 quire   20 quires = 1 ream

Measures of Time

60 seconds (sec.)	= 1 minute (min.)
60 minutes	= 1 hour (hr.)
24 hours	= 1 day (da.)
7 days	= 1 week (wk.)
12 months	= 1 year (yr.)
360 days	= 1 commercial year
365 days	= 1 common year (yr.)
366 days	= 1 leap year
10 years	= 1 decade
100 years	= 1 century

METRIC UNITS

Measures of Length

10 millimeters (mm.)	= 1 centimeter (cm.)
10 centimeters	= 1 decimeter (dm.)
10 decimeters	= 1 meter (m.)
1 meter	= 39.37 inches
1 yard	= .9144 meter

Measures of Surface

100 square millimeters (sq. mm.)	= 1 square centimeter (sq. cm.)
100 square centimeters	= 1 square decimeter (sq. dm.)
100 square decimeters	= 1 square meter (sq. m.)

Measures of Volume

1000 cubic millimeters (cu. mm.)	= 1 cubic centimeter (cu. cm.)
1000 cubic centimeters	= 1 cubic decimeter (cu. dm.)
1000 cubic decimeters	= 1 cubic meter (cu. m.)
1 cubic meter	= 1.308 cubic yards
1 cubic yard	= .765 cubic meter

Measures of Capacity

1 liter (l.)	= .908 dry quart
1 dry quart	= 1.1012 liters
1 liter	= 1.0567 liquid quarts
1 liquid quart	= .94636 liter

Measures of Weight

10 milligrams (mg.)	= 1 centigram (cg.)
10 centigrams	= 1 decigram (dg.)
10 decigrams	= 1 gram (g.)
10 grams	= 1 dekagram (Dg.)
10 dekagrams	= 1 hectogram (Hg.)
10 hectograms	= 1 kilogram (Kg.)
1 kilogram	= 2.2 pounds
1 gram	= weight of 1 cubic centimeter of water
	= .0022 pounds
1 pound	= 453.59 grams

Formulas

Area of a rectangle	$A=bh$
Area of a square	$A=s^2$
Area of a parallelogram	$A=bh$
Area of a triangle	$A=\frac{1}{2}bh$
Area of a trapezoid	$A=\frac{1}{2}(b_1+b_2)h$
Circumference of a circle	$c=\pi d$ or $2\pi r$
	where $\pi=\frac{22}{7}$ or 3.14159 approximately
Area of a circle	$A=\pi r^2$
Area of surface of a cylinder	$S=2\pi rh$
Area of surface of a cone	$S=\pi rl$
Area of surface of a sphere	$S=4\pi r^2$
Area of lateral surface of a right prism	$S=pe$
Area of lateral surface of a regular pyramid	$S=\frac{1}{2}ps$
Volume of a block	$V=lwh$
Volume of a cube	$V=e^3$
Volume of a prism	$V=bh$
Volume of a pyramid	$V=\frac{1}{3}bh$
Volume of a cylinder	$V=\pi r^2h$
Volume of a cone	$V=\frac{1}{3}\pi r^2h$
Volume of a sphere	$V=\frac{4}{3}\pi r^3$

INDEX

(References are to pages)